CALDERÓN–ZYGMUND ESTIMATES FOR HIGHER ORDER SYSTEMS WITH $p(x)$ GROWTH

JENS HABERMANN

ABSTRACT. For weak solutions $u \in W^{m,1}(\Omega;\mathbb{R}^N)$ of higher order systems of the type

$$\int_\Omega \langle A(x,D^m u), D^m\varphi\rangle\, dx = \int_\Omega \left\langle |F|^{p(x)-2}F, D^m\varphi\right\rangle dx, \quad \text{for all } \varphi \in C_c^\infty(\Omega;\mathbb{R}^N), \quad m>1,$$

with variable growth exponent $p:\Omega\to(1,\infty)$ we prove that if $|F|^{p(\cdot)} \in L^q_{loc}(\Omega)$ with $1<q<\frac{n}{n-2}+\delta$, then $|D^m u|^{p(\cdot)} \in L^q_{loc}(\Omega)$. We should note that we prove this implication both in the non–degenerate ($\mu>0$) and in the degenerate case ($\mu=0$).

1. Introduction

In this paper we are concerned with a regularity result for weak solutions of systems of higher order with $p(x)$– growth.

Let $n \in \mathbb{N}_{\geq 2}, N \in \mathbb{N}_{\geq 1}$ and $\Omega \subset \mathbb{R}^n$ a bounded domain. We consider weak solutions of the system

$$\int_\Omega \langle A\left(x,D^m u\right), D^m\varphi\rangle\, dx = \int_\Omega \left\langle |F(x)|^{p(x)-2}F(x), D^m\varphi\right\rangle dx, \tag{1.1}$$

for $\varphi \in W_0^{m,1}\left(\Omega;\mathbb{R}^N\right)$ with $|D^m\varphi|^{p(\cdot)} \in L^1_{loc}(\Omega)$, $\operatorname{supp}\varphi \Subset \Omega$. Here A denotes a vector field $A:\Omega\times\odot^m(\mathbb{R}^n,\mathbb{R}^N) \to \mathrm{Hom}(\odot^m(\mathbb{R}^n,\mathbb{R}^N),\mathbb{R})$, $F:\Omega\to\mathbb{R}^{N\binom{n+m-1}{m}}$, and $p:\Omega\to(1,\infty)$ a measurable function. $\odot^m(\mathbb{R}^n,\mathbb{R}^N)$ denotes the space of symmetric m– linear forms on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. The coefficient A is supposed to have $p(x)$– growth, i.e. for $\mu\in[0,1]$ there holds

$$\langle D_z A\,(x,z)\,\lambda,\lambda\rangle \approx \left(\mu^2+|z|^2\right)^{\frac{p(x)-2}{2}}|\lambda|^2.$$

Additionally we assume that the coefficient A is continuous with respect to the first variable and that there exists a modolus of continuity for the exponent function p, which satisfies

$$\lim_{\rho\downarrow 0}\omega(\rho)\log\left(\tfrac{1}{\rho}\right) = 0. \tag{1.2}$$

Solutions of systems of the type (1.1) with $p(x)$ growth can typically be shown to be an element of the Sobolev space $W^{m,p(\cdot)}_{loc}(\Omega;\mathbb{R}^n)$. See Definition 2.1 for an introduction to these spaces.

There have been many investigations on properties of such generalized function spaces in the last years. See for example [28], [13], [18], [11], [19], and especially [9] and [25] for properties of the maximal function on generalized Lebesgue spaces. We note that the linear counterpart to this paper, namely the generalization of the classical Calderón-Zygmund Theorem [5] to variable Lebesgue spaces has been done by Diening an Růžička in [12].

We show in this paper that there exists $\delta>0$ such that if $|F|^{p(\cdot)} \in L^q_{loc}(\Omega)$ with $1<q<\frac{n}{n-2}+\delta$, and u is a solution of system (1.1), then $|D^m u|^{p(\cdot)} \in L^q_{loc}(\Omega)$.

In the case of second order equations ($N=1$, $m=1$) and for second order systems with special structure, as for example the $p(x)$ Laplace system, such a result is proved in [3], without any restriction on q. This is due to the fact that in this special situation one can prove a L^∞ estimate for

the derivative Dw of the solution w of a suitable frozen problem. In the case of general systems (for the second order case see [21]) this is not possible. Nevertheless one obtains higher differentiability in the sense that $D^{m+1}w \in L^{\tilde{p}}$ with a suitable exponent $\tilde{p}$, depending on the exponent function p. This can be exploited to achieve the desired higher integrability in the sense of the above statement, with a restriction on the higher integrability exponent q.

The strategy of the proof in this paper follows in a certain sense the ideas in [3] and [21]. The key to the proof is an application of a Calderón–Zygmund type estimate on level sets of the maximal function of $|D^m u|^{p(\cdot)}$. Therefore the solution will be compared to the solution w of a problem, which is 'frozen' in a point x_M and therefore has the structure of a problem with constant growth exponent p_2. The solution w turns out to be higher differentiable, which translates via Sobolev–Poincaré's inequality into higher integrability of $|D^m w|^{p_2}$. By a suitable comparison estimate between $D^m u$ and $D^m w$, this carries over to the solution w.

One should note that we consider both the non degenerate ($\mu \neq 0$) and the degenerate ($\mu = 0$) elliptic case in this paper. Therefore the a priori estimates for the solution of the frozen problem are shown more or less in detail, especially pointing out the differences between the non degenerate and the degenerate case.

The author should mention that some parts of the proof (especially the comparison estimate) are widely similar to the proof in the second order case. Therefore at those points the estimates are shortened very much or cited from other papers. All of the statements are proved in a careful and extensive way in [16].

Acknowledgements. I would like to thank Prof. Dr. Giuseppe Mingione for his helpful comments and many fruitful discussions about systems and functionals with $p(x)$ growth.

2. Notations and Setting

We consider weak solutions of system (1.1). Note that, using multi indices, (1.1) reads as follows:

$$\int_\Omega \sum_{|\alpha|=m} A^i_\alpha(x, D^m u) D^\alpha \varphi_i \, dx = \int_\Omega \sum_{|\alpha|=m} |F(x)|^{p(x)-2} F^i_\alpha(x) D^\alpha \varphi_i \, dx, \quad i = 1, \dots, N.$$

Since the space $\odot^m(\mathbb{R}^n, \mathbb{R}^N)$ of symmetric m–linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^N$ can be identified with the space $\mathbb{R}^{N\binom{n+m-1}{m}}$, we consider A as a mapping $A : \Omega \times \mathbb{R}^{N\binom{n+m-1}{m}} \to \mathrm{Hom}(\mathbb{R}^{N\binom{n+m-1}{m}}, \mathbb{R})$. Additionally for the seek of brevity we introduce the abbreviation $\mathcal{N} \equiv N\binom{n+m-1}{m}$. Thus we have $D^m u(x) \in \mathbb{R}^{\mathcal{N}}$.

In the sequel we assume that the following structure conditions are satisfied: Concerning the vector field A we suppose the mapping $z \mapsto A(\cdot, z)$ to be of class $C^0(\mathbb{R}^{\mathcal{N}}) \cap C^1(\mathbb{R}^{\mathcal{N}}) \setminus \{0\})$ and to satisfy the following growth, ellipticity and continuity assumptions:

$$\nu \left(\mu^2 + |z|^2\right)^{\frac{p(x)-2}{2}} |\lambda|^2 \leq \langle D_z A\left(x, z\right) \lambda, \lambda \rangle \leq L \left(\mu^2 + |z|^2\right)^{\frac{p(x)-2}{2}} |\lambda|^2, \tag{2.1}$$

$$|A\left(x, z\right) - A\left(y, z\right)| \leq L\omega\left(|x-y|\right) \left[\left(\mu^2 + |z|^2\right)^{\frac{p(x)-1}{2}} + \left(\mu^2 + |z|^2\right)^{\frac{p(y)-1}{2}}\right] \left|\log\left(\mu^2 + |z|^2\right)\right|, \tag{2.2}$$

for all $x, y \in \Omega,\ z, \lambda \in \mathbb{R}^{\mathcal{N}}, z \neq 0$ where $\nu^{-1}, L \in [1, \infty),\ \mu \in [0, 1]$. The parameter μ is introduced in order to consider both, the degenerate and the non degenerate case. We assume that the modulus of continuity $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is a non decreasing, concave and continuous function satisfying $\omega(0) = 0$. For the function $p : \Omega \to (1, \infty)$ we assume that

$$1 < \gamma_1 \leq p(x) \leq \gamma_2 < +\infty, \tag{2.3}$$

for all $x \in \Omega$, as well as

$$|p(x) - p(y)| \leq \omega\left(|x-y|\right), \tag{2.4}$$

for all $x, y \in \Omega$, where ω is supposed to fulfill condition (1.2).

Remark. *By (2.1) we can assume that – eventually enlarging the constant L, reducing ν respectively – there holds:*

$$|A(x,z)| \le L\left(\mu^2 + |z|^2\right)^{(p(x)-1)/2}, \tag{2.5}$$

and

$$\nu\left(\mu^2 + |z|^2\right)^{p(x)/2} - L \le \langle A(x,z), z\rangle \qquad \text{for all} \;\; x \in \Omega, z \in \mathbb{R}^{\mathcal{N}}. \tag{2.6}$$

■

Definition 2.1 (Generalized Lebesgue and Sobolev spaces)**.** For a bounded domain $\Omega \subset \mathbb{R}^n$ and a measurable function $p : \Omega \to (1, \infty)$ we define the generalized Lebesgue space

$$L^{p(\cdot)}(\Omega;\mathbb{R}^N) \equiv \left\{ f \in L^1(\Omega;\mathbb{R}^N) : \int_\Omega |\lambda f(x)|^{p(x)}\, dx < \infty \quad \text{for some} \quad \lambda > 0 \right\},$$

which, endowed with the Luxembourg norm

$$||f||_{L^{p(\cdot)}(\Omega;\mathbb{R}^N)} \equiv \inf\left\{ \lambda > 0 : \int_\Omega \left|\tfrac{f(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}$$

becomes a Banach space. Furthermore the generalized Sobolev space is defined as

$$W^{m,p(\cdot)}(\Omega) \equiv \left\{ f \in L^{p(\cdot)}(\Omega) : \; D^\alpha f \in L^{p(\cdot)}(\Omega) \quad \text{for all} \quad 0 \le |\alpha| \le m \right\},$$

and also becomes a Banach space if endowed with the norm

$$||f||_{W^{m,p(\cdot)}(\Omega;\mathbb{R}^N)} \equiv \sum_{|\alpha| \le m} ||D^\alpha f||_{L^{p(\cdot)}(\Omega;\mathbb{R}^N)}.$$

See for example [28], [18], [9] and [13] for more details and further references on these spaces. ■

The main statement of this paper is the following

Theorem 2.2. *Let $u \in W^{m,p(\cdot)}(\Omega;\mathbb{R}^N)$ be a weak solution of system (1.1) under the growth, ellipticity and continuity assumptions (2.1) and (2.2) for the vector field A, condition (2.3) for the function p and the assumption (1.2) for the modulus of continuity of p. Then there exists $\delta \equiv \delta(n, m, \gamma_1, \gamma_2, L/\nu) > 0$ such that if $|F|^{p(\cdot)} \in L^q_{loc}(\Omega)$ for some exponent q, satisfying*

$$1 < q < \frac{n}{n-2} + \delta, \tag{2.7}$$

then

$$|D^m u|^{p(\cdot)} \in L^q_{loc}(\Omega). \tag{2.8}$$

In particular there holds: If $\Omega' \Subset \Omega$ and $|F|^{p(\cdot)} \in L^q(\Omega')$, then for every given $\varepsilon \in (0, q-1)$ there exists a positive radius $R_0 > 0$, depending on

$$n, N, m, \gamma_1, \gamma_2, \nu, L, \varepsilon, q, \omega(\cdot), \left\||D^m u|^{p(\cdot)}\right\|_{L^1(\Omega)}, \left\||F|^{p(\cdot)}\right\|_{L^q(\Omega')}$$

such that for any cube $Q_{4R} \Subset \Omega'$ and $R \le R_0$ there holds

$$\left(\fint_{Q_R} |D^m u|^{p(x)q}\, dx \right)^{1/q} \le cK^\varepsilon \fint_{Q_{4R}} |D^m u|^{p(x)}\, dx + cK^\varepsilon \left(\fint_{Q_{4R}} |F|^{p(x)q}\, dx + 1 \right)^{1/q},$$

where $c \equiv c(n, N, m, \gamma_1, \gamma_2, \nu, L, q)$ and

$$K := \int_{Q_{4R}} |D^m u|^{p(x)} + |F|^{p(x)(1+\varepsilon)}\, dx + 1.$$

■

Remarks on the notation. In the whole paper $\Omega \subset \mathbb{R}^n, (n \geq 2)$ denotes a bounded domain in the space $\mathbb{R}^n$ and $Q\,(x, R) \equiv Q_R(x)$ a cube whose axes are parallel to the axes of the coordinate system, with center x and side length $2R$. Sometimes R will also be called the 'radius' of the cube. The Lebesgue measure of a measurable set A is abbreviated by $|A| \equiv \mathcal{L}^n(A)$. For a locally integrable function $u \in L^1_{loc}(\Omega)$ we define the mean value on the cube Q by

$$(u)_{x_0,R} := \fint_{Q(x_0,R)} u(x)\,dx = \frac{1}{|Q(x_0,R)|}\int_{Q(x_0,R)} u(x)\,dx.$$

In the case the centre of the cube is obvious from the context, we will often just write Q_R or Q instead of $Q(x_0, R)$, $(u)_R$ instead of $(u)_{x_0,R}$ respectively.

The letter c denotes a constant which will not necessarily be the same at different places in the paper and which may sometimes change from line to line. Constants that will be referred to at other points of the work, will be signed in a unique way, mostly by different indices. In the case we want to emphasize the fact that a constant changes from one line to another, we will label this by mathematical accents, as for example $\tilde{c}$ or $\bar{c}$. For the survey we will not specify the dependencies of the constants in between the estimates, but of course at the end of them.

For $\Omega \subset \mathbb{R}^n, p > 1$, let $L^p(\Omega; \mathbb{R}^N)$ be the well known Lebesgue space to the power p. For $m \in \mathbb{N}$ we define the Sobolev space

$$W^{m,p}\left(\Omega; \mathbb{R}^N\right) := \left\{u \in L^p\left(\Omega, \mathbb{R}^N\right) : D^\alpha u \in L^p\left(\Omega\right) \text{ for } 0 \leq |\alpha| \leq m\right\},$$

with the multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and the abbreviations $|\alpha| := \alpha_1 + \ldots + \alpha_n$ and $D^\alpha u := D_1^{\alpha_1} \ldots D_n^{\alpha_n} u$. Furthermore let $W_0^{m,p}(\Omega; \mathbb{R}^N)$ denote the closure of $C^\infty(\Omega; \mathbb{R}^N)$ in the space $W^{m,p}(\Omega; \mathbb{R}^N)$.

3. A priori estimates

As we will see in the proof, the most important difficulty compared to the proof in the second order case ($m = 1$, see [3]) comes from the a priori estimate, which is different for higher order systems, since one can not obtain an $L^\infty-$ bound for the derivative $D^m u$. The optimal result is the following

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p > 1$ constant and $w \in W^{m,p}(\Omega; \mathbb{R}^N)$ a weak solution of the system*

$$\int_\Omega \langle A\left(D^m w\right), D^m \varphi\rangle\, dx = 0 \qquad \text{for all } \varphi \in W_0^{m,p}\left(\Omega; \mathbb{R}^N\right), \tag{3.1}$$

in which the function $A : \mathbb{R}^{\mathcal{N}} \to Hom(\mathbb{R}^{\mathcal{N}}, \mathbb{R})$ is of the class C^1 and satisfies the conditions

$$\begin{gathered} \nu\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2 \leq \langle DA(z)\lambda, \lambda\rangle \leq L\left(\mu^2 + |z|^2\right)^{\frac{p-2}{2}} |\lambda|^2, \\ |A(z)| \leq L\left(\mu^2 + |z|^2\right)^{\frac{p-1}{2}}, \end{gathered} \tag{3.2}$$

for all $z \in \mathbb{R}^{\mathcal{N}}$. Then the following holds:
In the case $1 < p < 2$ we have $w \in W^{m+1,p}_{loc}\left(\Omega; \mathbb{R}^N\right)$ together with the estimate

$$\int_{Q_{\tau R}} \left|D^{m+1} w\right|^p dx \leq \frac{c}{R^2}\int_{Q_R}\left(\mu^2 + |D^m w|^2\right)^{p/2} dx. \tag{3.3}$$

Additionally distinguishing the cases $\mu \neq 0$ and $\mu = 0$, we obtain furthermore

$$\int_{Q_{\tau R}} \left|D\left[\left(\mu^2 + |D^m w|^2\right)^{p/4}\right]\right|^2 dx \leq \frac{c}{R^2}\int_{Q_R}\left(\mu^2 + |D^m w|^2\right)^{p/2} dx \qquad (\mu \neq 0) \tag{3.4}$$

$$\int_{Q_{\tau R}} \left|D\left[|D^m w|^{\frac{p-2}{2}} D^m w\right]\right|^2 dx \leq \frac{c}{R^2}\int_{Q_R} |D^m w|^p\, dx \qquad (\mu = 0). \tag{3.5}$$

for any cube $Q_R \Subset \Omega$ and any $\tau \in (0,1)$.

In the case $p \geq 2$ we have $D\left[\left(\mu^2+|D^m w|^2\right)^{\frac{p-2}{4}} D^m w\right] \in L^2_{loc}(\Omega)$ and for any cube $Q_R \Subset \Omega$ and any $\tau \in (0,1)$ there holds

$$\int_{Q_{\tau R}} \left| D\left[\left(\mu^2+|D^m w|^2\right)^{\frac{p-2}{4}} D^m w\right]\right|^2 dx \leq \frac{c}{R^2}\int_{Q_R} \left(\mu^2+|D^m w|^2\right)^{p/2} dx. \tag{3.6}$$

Moreover the constants in the estimates above depend on n, N, m, p, τ and L/ν and $c \uparrow \infty$ as $\tau \uparrow 1$. ■

Proof. We start by proceeding analoguously to the second order case. For $h \in \mathbb{R}$ with $|h| < (1-\tau)R$ and $x \in Q_{\tau R}$ we denote by

$$\tau_{s,h} w(x) := w(x + he_s) - w(x)$$

the finite difference and by

$$\Delta_{s,h} w := \frac{\tau_{s,h} w}{h},$$

the difference quotient in direction e_s, where e_s denotes the sth unit vector in $\mathbb{R}^n$. We consider the test function

$$\varphi = \Delta_{s,-h}\left(\eta^{2m}\Delta_{s,h}(w-P)\right),$$

with a suitable polynomial P. Moreover for $R > 0$ and $0 < |h| < R$ we denote by

$$Q_{R,h} \equiv \{x \in Q_R : \ \operatorname{dist}(x, \partial Q_R) > |h|\}$$

the inner parallel cube, whose sidelength is $R - |h|$. For $w \in W^{m,p}(Q_R)$ we have $\Delta_{s,h} w \in W^{m,p}(Q_{R,h})$. In the definition of φ we choose $\eta \in C^\infty_c(Q_R)$ to be a standard cut-off function with the properties

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \ \text{ on } Q_{\tau R}, \quad \operatorname{spt}\eta \Subset Q_{\sqrt{\tau}R}$$

and

$$\left|D^k \eta\right| \leq \frac{c}{(R(\sqrt{\tau}-\tau))^k} = \frac{c(\tau)}{R^k}, \qquad \text{for} \quad k = 1, \ldots, m,$$

with $c(\tau) \to \infty$ as $\tau \nearrow 1$ or $\tau \searrow 0$. By the chain rule we immediately get

$$\left|D^k\left(\eta^{2m}\right)\right| \leq c(n,m,\tau)R^{-k}\sum_{j=1}^{k} \eta^{2m-j} \leq c(n,m,k,\tau)R^{-k}\eta^{2m-k}. \tag{3.7}$$

$P : \Omega \to \mathbb{R}^N$ denotes the unique polynomial of degree $m-1$ whose coefficients are chosen to satisfy

$$\left(D^k(w-P)\right)_{Q_{\sqrt{\tau}R}} = \fint_{Q_{\sqrt{\tau}R}} D^k(w-P)\, dx = 0 \quad \text{for} \ \ k = 0, \ldots, m-1. \tag{3.8}$$

Existence and uniqueness of such polynomials are well known and can be found for example in [14]. Testing (3.1), using standard identities for difference quotients and the general chain rule, we obtain for $|h| \leq R(1-\sqrt{\tau})$:

$$\begin{aligned} 0 &= \int_{Q_{\sqrt{\tau}R}} \left\langle \Delta_{s,h}A, \eta^{2m} D^m(\Delta_{s,h} w)\right\rangle dx \\ &\qquad + \int_{Q_{\sqrt{\tau}R}} \left\langle \Delta_{s,h}A, \sum_{k=1}^{m}\binom{m}{k} D^k\left(\eta^{2m}\right) \odot D^{m-k}\left(\Delta_{s,h}(w-P)\right)\right\rangle dx \\ &= I^{(1)} + I^{(2)}. \end{aligned}$$

Now we distinguish the cases $1 < p < 2$ and $p \geq 2$. We start with the case $p \geq 2$. We translate the growth and ellipticity conditions for A into conditions for $\Delta_{s,h}A$. Elementary calculations together with the differentiability of A show that

$$\Delta_{s,h}A(x) = B_h(x) D^m \Delta_{s,h} w(x), \tag{3.9}$$

with

$$B_h \equiv B_h(x) := \int_0^1 DA\left(D^m w(x) + tD^m \tau_{s,h} w(x)\right)\, dt.$$

Using (3.2) and the technical lemma 7.2 (with exponent $\frac{p-2}{2}$) we obtain the following pointwise estimates for B_h:

$$|B_h| \le c_1(p) L W_h^{p-2}, \qquad \langle B_h \lambda, \lambda\rangle \ge c_2(p)\nu W_h^{p-2}|\lambda|^2, \tag{3.10}$$

for $\lambda \in \mathbb{R}^{\mathcal{N}}$, with $W_h^2 := \mu^2 + |D^m w|^2 + |\tau_{s,h} D^m w|^2$.

Now, $I^{(1)}$ ist estimated by $(3.10)_2$ from below as follows:

$$I^{(1)} \ge c_2 \nu \int_{Q_{\sqrt{\tau}R}} \eta^{2m} W_h^{p-2} |\Delta_{s,h} D^m w|^2\, dx.$$

From (3.9) and (3.7) we infer that

$$I^{(2)} \le c \sum_{k=1}^m \int_{Q_{\sqrt{\tau}R}} |B_h|\, |\Delta_{s,h} D^m w|\, R^{-k} \eta^{2m-k} \left|\Delta_{s,h} D^{m-k}(w-P)\right|\, dx =: c\sum_{k=1}^m I_k^{(2)},$$

where $c \equiv c(n,m,\tau)$. $(3.10)_1$ and Young's inequality lead to

$$\begin{aligned} I_k^{(2)} \quad &\le \quad c_1 L\varepsilon \int_{Q_{\sqrt{\tau}R}} \eta^{2m} W_h^{p-2} |\Delta_{s,h} D^m w|^2\, dx \\ &\qquad + \frac{c_1 L}{4\varepsilon} \int_{Q_{\sqrt{\tau}R}} R^{-2k} \eta^{2(m-k)} W_h^{p-2} \left|\Delta_{s,h} D^{m-k}(w-P)\right|^2\, dx. \end{aligned}$$

Summing up the estimates for $I_k^{(2)}$, subsequently choosing $\varepsilon = \frac{c_2\nu}{2c_3 Lm}$ and taking into account that $\eta \equiv 1$ on $Q_{\tau R}$ finally leads to

$$\int_{Q_{\tau R}} W_h^{p-2} |\Delta_{s,h} D^m w|^2\, dx \le c \sum_{k=1}^m R^{-2k} \int_{Q_{\sqrt{\tau}R}} W_h^{p-2} \left|\Delta_{s,h} D^{m-k}(w-P)\right|^2\, dx, \tag{3.11}$$

where the constant c depends on n, m, p, τ and L/ν. Now we are going to estimate each of the terms appearing on the right hand side of (3.11). Writing $R^{-2k} = R^{-2(p-2)/p} \cdot R^{(2(1-k)p-4)/p}$, we obtain by Young's inequality (with exponents $p \equiv \frac{p}{p-2} > 1,\ q \equiv \frac{p}{2}$) for every $k = 1, \dots, m$

$$\begin{aligned} R^{-2k} &\int_{Q_{\sqrt{\tau}R}} W_h^{p-2} \left|\Delta_{s,h} D^{m-k}(w-P)\right|^2\, dx \\ &\le \tfrac{p-2}{p} R^{-2} \int_{Q_{\sqrt{\tau}R}} W_h^p\, dx + \tfrac{2}{p} R^{p(1-k)-2} \int_{Q_{\sqrt{\tau}R}} \left|\Delta_{s,h} D^{m-k}(w-P)\right|^p\, dx. \end{aligned}$$

Since $w \in W^{m,p}(Q_R)$, using standard estimates for difference quotients (note that $|h| \le R(1-\sqrt{\tau})$) and subsequently applying Poincaré's inequality $(k-1)$ times (note the choice of the polynomial P in (3.8)) provides for any $k = 1, \dots, m$:

$$\int_{Q_{\sqrt{\tau}R}} \left|\Delta_{s,h} D^{m-k}(w-P)\right|^p\, dx \le \int_{Q_R} \left|D^{m-k+1}(w-P)\right|^p\, dx \le cR^{(k-1)p} \int_{Q_R} |D^m w|^p\, dx,$$

which finally leads to

$$\int_{Q_{\tau R}} W_h^{p-2} |\Delta_{s,h} D^m w|^2\, dx \le c \sum_{k=1}^m \Big[R^{-2} \int_{Q_{\sqrt{\tau}R}} W_h^p\, dx + R^{-2} \int_{Q_R} |D^m w|^p\, dx\Big],$$

where the constant c depends only on n, N, m, p, τ and L/ν. Furthermore we easily see

$$\int_{Q_{\sqrt{\tau}R}} W_h^p\, dx \le c(p) \int_{Q_R} \left(\mu^2 + |D^m w|^2\right)^{p/2}\, dx.$$

Therefore we obtain for any $s = 1, \dots, n$ and $0 < |h| \leq R(1 - \sqrt{\tau})$ that

$$\int_{Q_{\tau R}} W_h^{p-2} \left|\Delta_{s,h} D^m w\right|^2 \, dx \leq cR^{-2} \int_{Q_R} \left(1 + |D^m w|^2\right)^{p/2} \, dx. \tag{3.12}$$

with $c \equiv c(n, N, m, p, \tau, L/\nu)$. In the case $1 < p < 2$ we proceed in a different way. However we note that the arguments which lead to the bound from below for $I^{(1)}$ in the case $p \geq 2$ also work here. Using $(3.10)_2$ we get

$$I^{(1)} \geq c_2 \nu \int_{Q_{\sqrt{\tau} R}} \eta^{2m} W_h^{p-2} \left|\Delta_{s,h} D^m w\right|^2 \, dx.$$

To treat $I^{(2)}$ we use a different formula for $\Delta_{s,h} A$. We do the following formal calculation, which holds for functions $w \in W^{m+1,p}$. The result for $w \in W^{m,p}$ can then be achieved by approximation. We write

$$\left[\Delta_{s,h} A(D^m w)\right](x) = \frac{1}{h} \int_0^1 \frac{d}{dt} A\left(D^m w\left(x + the_s\right)\right) \, dt = D_s \tilde{B}_h(x), \tag{3.13}$$

with

$$\tilde{B}_h(x) \equiv \int_0^1 A\left(D^m w\left(x + the_s\right)\right) \, dt.$$

By the growth condition (3.2) for A we find that

$$|\tilde{B}_h| \leq L \int_0^1 \left(\mu^2 + \left|D^m u\left(x + the_s\right)\right|^2\right)^{\frac{p-1}{2}} =: L \cdot Y_h.$$

For $I^{(2)}$ we write

$$I^{(2)} = c(m) \sum_{k=1}^{m} I_k^{(2)},$$

with

$$I_k^{(2)} \equiv \int_{Q_{\sqrt{\tau} R}} \left\langle D_s \tilde{B}_h, D^k\left(\eta^{2m}\right) \odot D^{m-k}\left(\Delta_{s,h}\left(w - P\right)\right)\right\rangle \, dx.$$

Taking into account (3.13) and $\operatorname{spt}\eta \Subset Q_{\sqrt{\tau} R}$ we obtain for $I_k^{(2)}$ by partial integration

$$\begin{aligned} I_k^{(2)} = &- \int_{Q_{\sqrt{\tau} R}} \left\langle \tilde{B}_h, D_s\left(D^k\left(\eta^{2m}\right)\right) D^{m-k}\left(\Delta_{s,h}\left(w - P\right)\right)\right\rangle \, dx \\ &- \int_{Q_{\sqrt{\tau} R}} \left\langle \tilde{B}_h, D^k\left(\eta^{2m}\right) D_s D^{m-k}\left(\Delta_{s,h}\left(w - P\right)\right)\right\rangle \, dx. \end{aligned}$$

By (3.7) we obtain

$$\left|D_s D^k\left(\eta^{2m}\right)\right| \leq \left|D^{k+1}\left(\eta^{2m}\right)\right| \leq c(n, m, k, \tau) R^{-(k+1)} \eta^{2m-(k+1)},$$

and therefore

$$\begin{aligned} |I_k^{(2)}| \leq\, & cR^{-(k+1)} \int_{Q_{\sqrt{\tau} R}} Y_h \eta^{2m-(k+1)} \left|D^{m-k} \Delta_{s,h}\left(w - P\right)\right| \, dx \\ &+ cR^{-k} \int_{Q_{\sqrt{\tau} R}} Y_h \eta^{2m-k} \left|D^{m-k+1} \Delta_{s,h}\left(w - P\right)\right| \, dx, \end{aligned}$$

with constants $c \equiv c(n,m,\tau)$. Combining the previous estimates we arrive at:

$$\begin{aligned}
c_2\nu \int_{Q_{\sqrt{\tau}R}} W_h^{p-2}\eta^{2m}\left|\Delta_{s,h}D^m w\right|^2\,dx &\le c\sum_{k=1}^{m} R^{-(k+1)}\int_{Q_{\sqrt{\tau}R}} Y_h\eta^{2m-(k+1)}\left|D^{m-k}\Delta_{s,h}(w-P)\right|\,dx\\
&\quad + c\sum_{k=1}^{m} R^{-k}\int_{Q_{\sqrt{\tau}R}} Y_h\eta^{2m-k}\left|D^{m-k+1}\Delta_{s,h}(w-P)\right|\,dx\\
&= cR^{-1}\int_{Q_{\sqrt{\tau}R}} Y_h\eta^{2m-1}\left|D^m\Delta_{s,h}w\right|\,dx\\
&\quad + c\sum_{k=1}^{m} R^{-(k+1)}\int_{Q_{\sqrt{\tau}R}} Y_h\eta^{2m-(k+1)}\left|D^{m-k}\Delta_{s,h}(w-P)\right|\,dx\\
&= I^{(3)} + \sum_{k=1}^{m} I_k^{(4)}.
\end{aligned}$$

We first consider $I^{(3)}$. Applying Young's inequality we get

$$\begin{aligned}
R^{-1}Y_h\eta^{2m-1}\left|D^m\Delta_{s,h}u\right| &= R^{-1}Y_h W_h^{\frac{2-p}{2}} W_h^{\frac{p-2}{2}}\eta^m\eta^{m-1}\left|D^m\Delta_{s,h}w\right|\\
&\le \varepsilon\eta^{2m}W_h^{p-2}\left|D^m\Delta_{s,h}w\right|^2 + \frac{1}{4\varepsilon}R^{-2}\eta^{2m-2}W_h^{2-p}Y_h^2.
\end{aligned}$$

By a suitable choice of ε, we can absorb the first term on the left hand side of (3.14). The second term can be estimated by Young's inequality (with exponents $p \equiv \frac{p}{2-p}$, $q \equiv \frac{p}{2(p-1)}$) as follows:

$$R^{-2}W_h^{2-p}Y_h^2 \le R^{-2}\left(\tfrac{2-p}{p}\,W_h^p + \tfrac{2(p-1)}{p}\,Y_h^{\frac{p}{p-1}}\right) \le c(p)R^{-2}\left(W_h^p + Y_h^{\frac{p}{p-1}}\right).$$

To estimate $I_k^{(4)}$ we write $R^{-(k+1)} = R^{2(1-p)/p}R^{(p(1-k)-2)/p}$ and obtain by Young's inequality

$$\begin{aligned}
I_k^{(4)} &= cR^{-(k+1)}\int_{Q_{\sqrt{\tau}R}} \eta^{2m-(k+1)}Y_h\left|D^{m-k}\Delta_{s,h}(w-P)\right|\,dx\\
&\le c\tfrac{p-1}{p}R^{-2}\int_{Q_{\sqrt{\tau}R}} \eta^{2m-(k+1)}Y_h^{\frac{p}{p-1}}\,dx\\
&\qquad + c\tfrac{1}{p}R^{p(1-k)-2}\int_{Q_{\sqrt{\tau}R}} \eta^{2m-(k+1)}\left|D^{m-k}\Delta_{s,h}(w-P)\right|^p\,dx\\
&\le cR^{-2}\int_{Q_{\sqrt{\tau}R}} Y_h^{\frac{p}{p-1}}\,dx + cR^{p(1-k)-2}\int_{Q_{\sqrt{\tau}R}} \left|D^{m-k}\Delta_{s,h}(w-P)\right|^p\,dx.
\end{aligned}$$

Combining the previous estimates we arrive at

$$\begin{aligned}
\int_{Q_{\tau R}} W_h^{p-2}\eta^{2m}\left|\Delta_{s,h}D^m w\right|^2\,dx &\le cR^{-2}\int_{Q_{\sqrt{\tau}R}} W_h^p\,dx + cR^{-2}\int_{Q_{\sqrt{\tau}R}} Y_h^{\frac{p}{p-1}}\,dx\\
&\quad + c\sum_{k=1}^{m} R^{p(1-k)-2}\int_{Q_{\sqrt{\tau}R}} \left|D^{m-k}\Delta_{s,h}(w-P)\right|^p\,dx,
\end{aligned}$$

with constants $c \equiv c(n,m,\tau,p,L/\nu)$. By Hölder's inequality and Fubini's theorem we see

$$\int_{Q_{\sqrt{\tau}R}} Y_h^{\frac{p}{p-1}}\,dx \le \int_{Q_{\sqrt{\tau}R}}\int_0^1 \left(\mu^2 + \left|D^m w\,(x+the_s)\right|^2\right)^{p/2}\,dt\,dx \le \int_{Q_R}\left(\mu^2 + \left|D^m w(x)\right|^2\right)^{p/2}\,dx.$$

As in the case $p \ge 2$ we deduce easily

$$\int_{Q_{\sqrt{\tau}R}} W_h^p\,dx \le c(p)\int_{Q_R}\left(\mu^2 + \left|D^m w\right|^2\right)^{p/2}\,dx.$$

Furthermore again by standard estimates for difference quotients we obtain

$$\int_{Q_{\sqrt{\tau}R}} \left|D^{m-k}\Delta_{s,h}(w-P)\right|^p dx \le c(n,N)\int_{Q_R}\left|D^{m-k+1}(w-P)\right| dx.$$

Inserting this above we find that (note that $\eta \equiv 1$ on $Q_{\tau R}$)

$$\begin{aligned}\int_{Q_{\tau R}} W_h^{p-2}\left|\Delta_{s,h}D^m w\right|^2 dx \quad \le \quad & cR^{-2}\int_{Q_R}\left(\mu^2+|D^m w|^2\right)^{p/2} dx \\ & +c\sum_{k=1}^{m} R^{p(1-k)-2}\int_{Q_R}\left|D^{m-k+1}(w-P)\right|^p dx,\end{aligned}$$

with $c \equiv c(n,N,m,p,\tau,L/\nu)$. By the choice of the polynomial P (see (3.8)) we can apply $(k-1)$ times Poincaré's inequality to the integrals $\int_{Q_R}|D^{m-k+1}(w-P)|^p\,dx$; actually we have that

$$\int_{Q_{\sqrt{\tau}R}}\left|D^{m-k+1}(w-P)\right|^p dx \le c(n,N)R^{(k-1)p}\int_{Q_{\sqrt{\tau}R}}|D^m w|^p\,dx,$$

for all $k = 2,\dots,m$. Therefore we obtain

$$\begin{aligned}\int_{Q_{\tau R}} W_h^{p-2}\left|\Delta_{s,h}D^m w\right|^2 dx \quad & \le \quad cR^{-2}\int_{Q_R}\left(\mu^2+|D^m w|^2\right)^{p/2} dx + c\sum_{k=1}^{m}R^{-2}\int_{Q_{\sqrt{\tau}R}}|D^m w|^p\,dx \\ & \le \quad cR^{-2}\int_{Q_R}\left(\mu^2+|D^m w|^2\right)^{p/2} dx,\end{aligned}$$

with $c \equiv c(n,N,m,\tau,p,L/\nu)$. Hence for any $p>1$ there holds

$$\int_{Q_{\tau R}} W_h^{p-2}\left|\Delta_{s,h}D^m w\right|^2 dx \le cR^{-2}\int_{Q_R}\left(\mu^2+|D^m w|^2\right)^{p/2} dx, \tag{3.14}$$

with a constant $c \equiv c(n,N,m,p,\tau,L/\nu)$.

Now we distinguish the cases $1<p<2$ and $p \ge 2$:

The case $1<p<2$: We set $2\alpha := p\,(2-p)$, obtaining by Young's inequality

$$\left|\Delta_{s,h}D^m w\right|^p = W_h^{\alpha}W_h^{-\alpha}\left|\Delta_{s,h}D^m w\right|^p \le c(p)\left(W_h^p + W_h^{p-2}\left|\Delta_{s,h}D^m w\right|^2\right),$$

and therefore

$$\int_{Q_{\tau R}}\left|\Delta_{s,h}D^m w\right|^p dx \le cR^{-2}\int_{Q_R}\left(\mu^2+|D^m w|^2\right)^{p/2} dx. \tag{3.15}$$

By (3.15), we see that the sequence $\Delta_{s,h}D^m w$ is uniformly bounded in $L^p(Q_{\tau R})$. Therefore $\Delta_{s,h}D^m w$ converges as $h \to 0$ strongly in $L^p_{loc}(Q_{\tau R})$ to $D_s D^m w$, i.e. $w \in W^{m+1,p}_{loc}(Q_{\tau R})$ and (3.3) holds. On the other hand, for $\mu \in (0,1]$, a subsequence of $\Delta_{s,h}D^m w$ converges pointwise a.e. to $D_s D^m w$ as $h \to \infty$. With the convergence

$$\tau_{s,h}D^m w \stackrel{h\to 0}{\longrightarrow} 0 \quad \text{ in } L^p(Q_{\tau R}),$$

there holds

$$W_h = \left(\mu^2+|D^m w|^2+|\tau_{s,h}D^m w|^2\right)^{1/2} \stackrel{h\to 0}{\longrightarrow} \left(\mu^2+|D^m w|^2\right)^{1/2} \quad \text{ in } L^p(Q_{\tau R}).$$

This implies the pointwise almost everywhere convergence of a subsequence of W_h to $(\mu^2 + |D^m w|^2)^{1/2}$. By Fatou's Lemma we now conclude with (3.14):

$$\int_{Q_{\tau R}}(\mu^2+|D^m w|^2)^{\frac{p-2}{2}}|D_s D^m w|^2\,dx \le \frac{c}{R^2}\int_{Q_R}(\mu^2+|D^m w|^2)^{p/2}\,dx.$$

Therefore we end up with

$$\int_{Q_{\tau R}} \left(\mu^2 + |D^m w|^2\right)^{\frac{p-2}{2}} |D^{m+1} w|^2 \, dx \le \frac{c}{R^2} \int_{Q_R} \left(\mu^2 + |D^m w|^2\right)^{p/2} \, dx. \tag{3.16}$$

On the other hand, by differentiating, we have the estimate

$$\left| D\left[\left(\mu^2 + |D^m w|^2\right)^{p/4} \right] \right|^2 \le c \left(\mu^2 + |D^m w|^2\right)^{\frac{p-2}{2}} |D^{m+1} w|^2. \tag{3.17}$$

Therefore we conclude

$$\int_{Q_{\tau R}} \left| D\left[\left(\mu^2 + |D^m w|^2\right)^{p/4} \right] \right|^2 dx \le \frac{c}{R^2} \int_{Q_R} \left(\mu^2 + |D^m w|^2\right)^{p/2} \, dx,$$

which is exactly (3.4). We use an elementary algebraic property of the function $V_\mu(z) := (\mu^2 + |z|^2)^{(p-2)/4} z$ (see (8.45) on page 20) to obtain

$$\begin{aligned} &\left| \tau_{s,h} \left[\left(\mu^2 + |D^m w|^2\right)^{\frac{p-2}{4}} D^m w \right] \right|^2 \\ &\quad \le c \left[\mu^2 + |D^m w(x)|^2 + |D^m w(x + h e_s)|^2 \right]^{\frac{p-2}{2}} |D^m w(x + h e_s) - D^m w(x)|^2 \\ &\quad \le c \left[\mu^2 + |D^m w|^2 + |\tau_{s,h} D^m w|^2 \right]^{\frac{p-2}{2}} |\tau_{s,h} D^m w|^2 . \end{aligned} \tag{3.18}$$

Combining this estimate for $\mu = 0$ with (3.14), we end up with

$$\int_{Q_{\tau R}} \left| \Delta_{s,h} \left[|D^m w|^{\frac{p-2}{2}} D^m w \right] \right|^2 \le \frac{c}{R^2} \int_{Q_R} |D^m w|^p \, dx.$$

We see that the sequence $\Delta_{s,h}[|D^m w|^{\frac{p-2}{2}} D^m w]$ is uniformly bounded in $L^2(Q_{\tau R})$. By a standard lemma about difference quotients it converges as $h \to \infty$ strongly in $L^2_{loc}(Q_{\tau R})$ to $D_s[|D^m w|^{\frac{p-2}{2}} D^m w]$. The estimate above together with the convergence yield the desired estimate (3.5).

The case $p \ge 2$: We take (3.18) together with (3.14) to conclude

$$\int_{Q_{\tau R}} \left| \Delta_{s,h} \left[\left(\mu^2 + |D^m w|^2\right)^{\frac{p-2}{4}} D^m w \right] \right|^2 \le c R^{-2} \int_{Q_R} \left(\mu^2 + |D^m w|^2\right)^{p/2} \, dx. \tag{3.19}$$

By (3.19) we see that the sequence $\Delta_{s,h}[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w]$ is uniformly bounded in $L^2(Q_{\tau R})$ and therefore the sequence converges as $h \to 0$ strongly in $L^2_{loc}(Q_{\tau R})$ to $D_s[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w]$. The estimates above together with the convergence provide the desired estimate (3.6). □

4. An additional Gehring improvement

Starting by Lemma 3.1, we can now achieve by standard techniques a further higher integrability exponent in the following sense:

Lemma 4.1. *Let $w \in W^{m,p}(\Omega; \mathbb{R}^N)$ be a solution of the system* (3.1)*, which satisfies the structure conditions* (3.2)*. Then there exists $\delta \equiv \delta(n, m, p, L/\nu)$ and a constant $c \equiv c(n, m, p, L, M)$ such that the following holds:*

In the case $1 < p < 2$ and $\mu \neq 0$ we have the estimate

$$\int_{Q_{\tau R}} \left| D\left[(\mu^2 + |D^m w|^2)^{p/4} \right] \right|^{2(1+\delta)} dx \le \left(\frac{c}{R^2} \int_{Q_R} (\mu^2 + |D^m w|^2)^{p/2} \, dx \right)^{1+\delta}. \tag{4.1}$$

In the case $1 < p < 2$ and $\mu = 0$ we obtain

$$\int_{Q_{\tau R}} \left| D\left[|D^m w|^{\frac{p-2}{2}} D^m w \right] \right|^{2(1+\delta)} dx \le \left(\frac{c}{R^2} \int_{Q_R} |D^m w|^p \, dx \right)^{1+\delta}. \tag{4.2}$$

In the case $p \geq 2$ there holds for any $\mu \in [0,1]$:

$$\int_{Q_{\tau R}} \left| D\left[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w\right]\right|^{2(1+\delta)} dx \leq \left(\frac{c}{R^2}\int_{Q_R} (\mu^2 + |D^m w|^2)^{p/2}\, dx\right)^{1+\delta}. \tag{4.3}$$

■

Proof. Since such a result is more or less standard, we only show the main ideas of the proof here. Our aim is to show a reverse Hölder inequality which translates via Gehring's lemma into the desired higher integrability result. Distinguishing both the cases $p \geq 2$ and $p < 2$ and $\mu \neq 0$, $\mu = 0$, we proceed as follows: As in the proof of Lemma 3.1, we test system (3.1) with the function $\varphi \equiv D(\eta^{2m} D(w - P))$ with a suitable cut off function η and a polynomial of degree m which we specify later.

In the case $p \geq 2$, following the estimates in the proof of Lemma 3.1, see (3.11), finally applying Young's inequality, we obtain

$$\begin{aligned}
&\int_{Q_{\tau R}} (\mu^2 + |D^m w|^2)^{\frac{p-2}{2}} |D^{m+1} w|^2\, dx \\
&\qquad \leq c\left[\int_{Q_{\sqrt{\tau}R}} (\mu^2 + |D^m w|^2)^{p/2}\, dx + \sum_{k=1}^{m} \int_{Q_{\sqrt{\tau}R}} R^{-kp} |D^{m+1-k}(w-P)|^p\, dx\right].
\end{aligned}$$

Having in mind (3.18), the left hand side is estimated from below by

$$\int_{Q_{\tau R}} \left| D\left[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w\right]\right|^2 dx.$$

Choosing the coefficients of the polynomial P in such a way that

$$\int_{Q_{\sqrt{\tau}R}} D^j (w-P)\, dx = 0, \qquad \text{for } \ j = 0, \ldots, m-1\,,$$

we can use Poincaré's inequality and elementary algebraic calculations to estimate the second term of the right hand side from above by

$$cR^{-2} \int_{Q_{\sqrt{\tau}R}} (\mu^2 + |D^m (w-P)|^2)^{\frac{p-2}{2}} |D^m (w-P)|^2\, dx.$$

Now choosing the highest order coefficients of the polynomial P such that

$$\begin{aligned}
&\int_{Q_{\sqrt{\tau}R}} (\mu^2 + |D^m (w-P)|^2)^{\frac{p-2}{2}} |D^m (w-P)|^2\, dx \\
&\qquad = \int_{Q_{\sqrt{\tau}R}} \left| (\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w - \Big((\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w\Big)_{Q_{\sqrt{\tau}R}} \right|^2 dx,
\end{aligned}$$

we can apply Sobolev-Poincaré's inequality to conclude a reverse Hölder inequality of the type

$$\begin{aligned}
&\int_{Q_{\tau R}} \left| D\left[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}} D^m w\right]\right|^2 dx \\
&\qquad \leq c\left[\left(\int_{Q_{\sqrt{\tau}R}} \left| D\left[(\mu^2 + |D^m w|^2)^{\frac{p-2}{4}}) D^m w\right]\right|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} + R^{-2}\int_{Q_{\sqrt{\tau}R}} (\mu^2 + |D^m w|^2)^{p/2}\, dx\right].
\end{aligned}$$

Gehring's Lemma now provides the desired higher integrability. Combining this result with the estimates in Lemma (3.1), we end up with (4.3).

Remark. Here we also need higher integrability of $|D^m w|^p$, which is standard to prove. See for example [15] for higher integrability results of this type.

In the case $1 < p < 2$ and $\mu \neq 0$, we finally obtain, using the same test function as above (see (3.14)) with a polynomial P of degree $m-1$ and taking into consideration (3.17)

$$\begin{aligned}\int_{Q_{\tau R}} \left|D\left[(\mu^2+|D^m w|^2)^{p/4}\right]\right|^2 dx \\ \leq \frac{c}{R^2}\Bigg[\int_{Q_{\sqrt{\tau}R}} \left|(\mu^2+|D^m w|^2)^{p/4} - \left((\mu^2+|D^m w|^2)^{p/4}\right)_{Q_{\sqrt{\tau}R}}\right|^2 dx \\ + \int_{Q_{\sqrt{\tau}R}} (\mu^2+|D^m w|^2)^{p/2}\, dx\Bigg].\end{aligned}$$

We now apply Sobolev-Poincaré's inequality, obtaining a reverse Hölder inequality of which allows us to apply Sobolev-Poincaré's inequality, obtaining a reverse Hölder inequality of the type

$$\begin{aligned}\int_{Q_{\tau R}} \left|D\left[(\mu^2+|D^m w|^2)^{p/4}\right]\right|^2 dx \\ \leq c\left[\left(\int_{Q_{\sqrt{\tau}R}} \left|D\left[(\mu^2+|D^m w|^2)^{p/4}\right]\right|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} + R^{-2}\int_{Q_{\sqrt{\tau}R}} (\mu^2+|D^m w|^2)^{p/2}\, dx\right].\end{aligned}$$

Again applying Gehrings Lemma and combining the result with the estimate of Lemma 3.1 and a priori higher integrability for $|D^m w|^p$, provides the desired estimate (4.1). In the case $\mu = 0$ we have by (3.14) and (3.18)

$$\begin{aligned}\int_{Q_{\tau R}} \left|D\left[|D^m w|^{\frac{p-2}{2}} D^m w\right]\right|^2 dx \\ \leq \frac{c}{R^2}\left[\int_{Q_{\sqrt{\tau}R}} \left||D^m w|^{\frac{p-2}{2}} D^m w - \left(|D^m w|^{\frac{p-2}{2}} D^m w\right)_{Q_{\sqrt{\tau}R}}\right|^2 dx + \int_{Q_{\sqrt{\tau}R}} |D^m w|^p\, dx\right].\end{aligned}$$

Sobolev-Poincaré's inequality now provides

$$\begin{aligned}\int_{Q_{\tau R}} \left|D\left[|D^m w|^{\frac{p-2}{2}} D^m w\right]\right|^2 dx \\ \leq c\left[\left(\int_{Q_{\tau R}} \left|D\left[|D^m w|^{\frac{p-2}{2}} D^m w\right]\right|^{\frac{2n}{n+2}} dx\right)^{\frac{n+2}{n}} + R^{-2}\int_{Q_{\sqrt{\tau}R}} |D^m w|^p\, dx\right].\end{aligned}$$

Again Gehrings Lemma and finally the estimates of Lemma 3.1 lead to the desired estimate (4.2). □

5. Calderón–Zygmund coverings

We consider a cube $Q_0 \subset \mathbb{R}^n$ and define by $\mathcal{D}(Q_0)$ the set of all dyadic subcubes Q of Q_0, i.e. those cubes with sides parallel to the sides of Q_0 that can be obtained from Q_0 by a positive finite number of dyadic subdivisions. We call Q_p *a predecessor* of Q, if Q is obtained from Q_p by a finite number of dyadic subdivisions. In particular we call $\tilde{Q} \in \mathcal{D}(Q_0)$ *the predecessor* of Q, if Q is obtained from $\tilde{Q}$ by exactly one dyadic subdivision from $\tilde{Q}$.

The following lemma will play an essential role in the proof Theorem 2.2. The proof is done by Calderón–Zygmund coverings and can be found for example in [4].

Lemma 5.1. *Let $Q_0 \subset \mathbb{R}^n$ be a cube. Moreover let $X \subset Y \subset Q_0$ be measurable sets satisfying the following: There exists $\delta > 0$ such that*

(i)
$$|X| < \delta |Q_0|,$$

and

(ii) *for any cube $Q \in \mathcal{D}(Q_0)$ there holds*

$$|X \cap Q| > \delta |Q| \Longrightarrow \tilde{Q} \subset Y,$$

in which $\tilde{Q}$ denotes the predecessor of Q.

Then there holds

$$|X| < \delta |Y|.$$

■

6. Hardy Littlewood maximal function

We will use properties of the Hardy Littlewood maximal function, which we will state here without proving them. For a more detailed discussion about maximal operators see [20] and [29].

Let $Q_0 \subset \mathbb{R}^n$ be a cube. For a function $f \in L^1(Q_0)$ we define the restricted maximal function on Q_0 by

$$M^*_{Q_0}(f)(x) := \sup_{Q \subseteq Q_0, x \in Q} ⨍_Q |f(y)| \, dy, \quad x \in Q_0, \tag{6.1}$$

where Q denotes an arbitrary subcube of Q_0, not necessarily centered in $x \in Q_0$. In an analogue way we define for $s > 1$ and $f \in L^s(Q_0)$

$$M^*_{s,Q_0}(f)(x) := \sup_{Q \subseteq Q_0, x \in Q} \left(⨍_Q |f(y)|^s \, dy \right)^{1/s}. \tag{6.2}$$

We will need the following properties of the maximal function operator:

Lemma 6.1. *For $Q_0 \subset \mathbb{R}^n$ and $s > 1$ let the maximal functions $M^*_{Q_0}$ and M^*_{s,Q_0} be defined as above. Then the following estimates hold:*

(M1) *For $f \in L^1(Q_0)$ and for any $\alpha > 0$ there holds*

$$\left| \left\{ x : M^*_{Q_0}(f)(x) > \alpha \right\} \right| \leq \frac{c_W}{\alpha} \int_{Q_0} |f(y)| \, dy, \tag{6.3}$$

with a constant $c_W \equiv c_W(n)$, for example $c_W = 3^n$ sufficies.

(M2) *For $f \in L^p(Q_0), 1 < p < \infty$ we have $M^*(f) \in L^p$ and there holds*

$$\int_{Q_0} \left| M^*_{Q_0}(f)(y) \right|^p \, dy \leq \frac{3^n e p^2}{p-1} \int_{Q_0} |f(y)|^p \, dy. \tag{6.4}$$

(M3) *An inequality similar to the one in (M2) holds also for the maximal function M^*_{s,Q_0}, i.e. for $p > s$ we have:*

$$\int_{Q_0} \left| M^*_{s,Q_0}(f)(y) \right|^p \, dy \leq \frac{3^n e p^2}{s(p-s)} \int_{Q_0} |f(y)|^p \, dy. \tag{6.5}$$

■

A direct consequence of (M1) is the following

Corollary 6.2. *Let $f \in L^s(Q_0)$ with $s > 1$. Then*

$$\left| \left\{ x : M^*_{Q_0}(f)(x) > \alpha \right\} \right| \leq \frac{c_W}{\alpha^s} \int_{Q_0} |f(y)|^s \, dy,$$

with the constant c_W of (M1). ■

7. Some technicalities

The following technical lemmas will be used at several points in the proof of the main theorem. Since they are more or less standard we will only cite them.

Lemma 7.1 ([10], Lemma 2.2)**.** *Let $p > 1$. Then there exists a constant c, such that for every $\mu \geq 0$, $\xi, \eta \in \mathbb{R}^k$ there holds*

$$(\mu^2 + |\xi|^2)^{p/2} \leq c(\mu^2 + |\eta|^2)^{p/2} + c(\mu^2 + |\xi|^2 + |\eta|^2)^{(p-2)/2}|\xi - \eta|^2.$$

■

Lemma 7.2 (see [6])**.** *Let $a, b \in \mathbb{R}^N$ and $\nu > -1$. Then there exist constants $c(\nu)$, $C(\nu) > 0$, such that there holds*

$$c(\nu)\left(\mu + |a| + |b|\right)^{\nu} \leq \int_0^1 \left(\mu + |a + tb|\right)^{\nu}\, dt \leq C(\nu)\left(\mu + |a| + |b|\right)^{\nu}.$$

■

8. Proof of the main theorem

The proof of Theorem 2.2 is at many stages similar or identical to the proof in the case $m = 1$, which is done in [3]. Therefore some of the estimates will only be cited (for example the comparison estimate). We will especially point out the differences to the higher order case here.

8.1. **Choice of constants and radii (I).** To proceed with the proof of Theorem 2.2 we initially fix some of the constants. The proof will take place on the cube $Q_{4R_0} \Subset \Omega$. The radius R_0 of this cube will be restricted at several points in the course of the proof. At first we choose the radius so small that

$$\begin{cases} \omega\left(8nR_0\right) \leq \sqrt{\frac{n+1}{n}} - 1, \\ 0 < \omega(R)\log\left(\frac{1}{R}\right) \leq \tilde{L}, \qquad \text{for any } 0 < R \leq 8nR_0. \end{cases} \tag{8.1}$$

Furthermore we set

$$K_0 := \int_{Q_{4R_0}} |D^m u|^{p(x)}\, dx + 1. \tag{8.2}$$

8.2. **Higher integrability.** We will show that the condition

$$\lim_{\rho \downarrow 0} \omega(R)\log\left(\frac{1}{\rho}\right) \leq M < +\infty, \tag{8.3}$$

on the modulus of continuity ω yields a certain higher integrability for $|D^m u|^{p(\cdot)}$. We note that condition (8.3) is weaker than condition (1.2) which is needed for proving the main theorem. Our result is the following

Lemma 8.1 (Higher integrability of $|D^m u|^{p(\cdot)}$)**.** *Let $u \in W^{m,1}\left(\Omega; \mathbb{R}^N\right)$ with $|D^m u|^{p(\cdot)} \in L^1_{loc}(\Omega)$ be a weak solution of (1.1) under the conditions (2.3), (8.3) as well as*

$$|A(x, z)| \leq L\left(1 + |z|^2\right)^{(p(x)-1)/2}, \tag{8.4}$$

and

$$\nu\left(\mu^2 + |z|^2\right)^{p(x)/2} - L \leq \langle A(x, z), z\rangle, \tag{8.5}$$

for all $x \in \Omega, z \in \mathbb{R}^{\mathcal{N}}$. Moreover let $F \in L^{p(\cdot)q}(\Omega;\mathbb{R}^{\mathcal{N}})$ for some $q > 1$. Then there exist constants $c \equiv c(n, N, m, \gamma_1, \gamma_2, L/\nu, \tilde{L}, M)$ and $c_g \equiv c_g(n, N, m, \gamma_1, \gamma_2, \nu, L, \tilde{L})$ such that the following holds: If R_0 is the radius from (8.1), K_0 from (8.2), $Q_{4R_0} \Subset \Omega$, $\sigma > 0$ a constant with

$$0 < \sigma \leq \sigma_0 := \min\left\{ \frac{c_g}{K_0^{\frac{2q\omega(8nR_0)}{\gamma_1}}},\ q-1,\ 1 \right\},$$

then for every cube $Q_R \subseteq Q_{4R_0}$ we have that

$$\left(\fint_{Q_{R/2}} |D^m u|^{p(x)(1+\sigma)}\, dx \right)^{\frac{1}{1+\sigma}} \leq c \fint_{Q_R} |D^m u|^{p(x)}\, dx + c\left(\fint_{Q_R} |F|^{p(x)(1+\sigma)}\, dx + 1 \right)^{\frac{1}{1+\sigma}}. \tag{8.6}$$

■

Proof. Since the proof of this result is in many points similar to the proof in the case $m = 1$, we only show the main steps here. Let $Q_R \subseteq Q_{4R_0}$ be a cube and

$$p_1 := \inf\{p(x) : x \in Q_R\}, \qquad p_2 := \sup\{p(x) : x \in Q_R\}. \tag{8.7}$$

Then, $p_2 - p_1 \leq \omega(2nR)$ and by the choice of R_0 in (8.1) we have

$$\frac{p_2}{p_1} = \frac{p_2 - p_1}{p_1} + 1 \leq \sqrt{\frac{n+1}{n}} =: \tilde{s}. \tag{8.8}$$

We test system (1.1) by the function $\varphi_1 \equiv \eta^{mp_2}(u - P)$, where $\eta \in C_c^\infty(Q_R)$ denotes a standard cut-off function satisfying $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $Q_{R/2}$ as well as $\left|D^k \eta\right| \leq \frac{1}{R^k}$ for $k = 1, \ldots, m$ and $P : \mathbb{R}^n \to \mathbb{R}$ denotes the unique polynomial of degree $m-1$ satisfying

$$\left(D^k(u - P)\right)_{Q_R} = \fint_{Q_R} D^k(u - P)\, dx = 0 \quad \text{for} \quad k = 0, \ldots, m-1. \tag{8.9}$$

It is easy to see that we have

$$\left|D^k(\eta^{mp_2})\right| \leq C(n, m, \gamma_2) R^{-k} \sum_{j=1}^{k} \eta^{mp_2 - j} \leq C(n, m, k, \gamma_2) R^{-k} \eta^{mp_2 - k}. \tag{8.10}$$

Setting in the test function and using (8.5) we obtain

$$\begin{aligned} \nu \int_{Q_R} \eta^{mp_2} |D^m u|^{p(x)}\, dx - L \quad &\leq \quad \int_{Q_R} \langle A(x, D^m u), D^m \varphi_1 \rangle\, dx \\ &\qquad - \int_{Q_R} \sum_{k=1}^{m} \binom{m}{k} \left\langle A(x, D^m u), D^k(\eta^{mp_2}) \odot D^{m-k}(u - P) \right\rangle dx \\ &= \quad I_1 + I_2, \end{aligned}$$

with the obvious labelling. Using (1.1), I_1 can be estimated by applying Young's inequality several times in a standard way (note that the constant in Young's inequality may depend on $p(x)$; writing the constant down explicitely, one can easily see that it can be estimated by a constant depending only on γ_1 and γ_2) and using the fact that $p(x) \leq p_2$ on Q_R to obtain

$$\begin{aligned} I_1 \leq{} & \varepsilon \int_{Q_R} \eta^{mp_2} |D^m u|^{p(x)}\, dx \\ & + c\left[\int_{Q_R} |F(x)|^{p(x)}\, dx + \sum_{k=1}^{m} \int_{Q_R} \left(\frac{\left|D^{m-k}(u - P)\right|^{p_2}}{R^{kp_2}} + 1 \right) dx \right], \end{aligned} \tag{8.11}$$

where $c \equiv c(m, \varepsilon, \gamma_1, \gamma_2)$. By (8.4), (8.10) and Young's inequality we estimate

$$I_2 \leq cL \sum_{k=1}^{m} \int_{Q_R} \left(1 + |D^m u|^{p(x)-1} \right) \eta^{mp_2 - k} \frac{|D^{m-k}(u - P)|}{R^k}\, dx,$$

where $c \equiv c(n, m, \gamma_2)$. We rewrite the integral appearing on the right hand side as follows :

$$\begin{aligned}
&\int_{Q_R} \left(1 + |D^m u|^{p(x)-1}\right) \eta^{mp_2-k} \frac{|D^{m-k}(u-P)|}{R^k}\, dx \\
&\qquad = \int_{Q_R} |D^m u|^{p(x)-1} \eta^{mp_2-k} \frac{|D^{m-k}(u-P)|}{R^k}\, dx + \int_{Q_R} \eta^{mp_2-k} \frac{|D^{m-k}(u-P)|}{R^k}\, dx \\
&\qquad = I_{2,1}^{(k)} + I_{2,2}^{(k)}.
\end{aligned}$$

The second integral we treat as usual noting that

$$I_{2,2}^{(k)} \le \int_{Q_R} \frac{|D^{m-k}(u-P)|^{p_2}}{R^{kp_2}} + 1\, dx.$$

Therefore it remains to get a bound for $I_{2,1}^{(k)}$. Since $\eta \le 1$ we have $\eta^{mp_2-k} \le \eta^{m(p_2-1)}$ for $k = 1, \ldots, m$, and hence by Young's inequality and $\frac{p(x)}{p(x)-1} \ge \frac{p_2}{p_2-1}$ we obtain that

$$I_{2,1}^{(k)} \le \varepsilon \int_{Q_R} \eta^{mp_2} |D^m u|^{p(x)}\, dx + c \int_{Q_R} \frac{|D^{m-k}(u-P)|^{p_2}}{R^{kp_2}} + 1\, dx.$$

Combining the estimates for $I_{2,1}^{(k)}$ and $I_{2,2}^{(k)}$ we finally arrive at

$$\begin{aligned}
\nu \int_{Q_R} \eta^{mp_2} |D^m u|^{p(x)}\, dx \quad &\le \quad cL\varepsilon \int_{Q_R} \eta^{mp_2} |D^m u|^{p(x)}\, dx \\
&\quad + cL \left[\int_{Q_R} |F(x)|^{p(x)}\, dx + \sum_{k=1}^{m} \int_{Q_R} \left(\frac{|D^{m-k}(u-P)|^{p_2}}{R^{kp_2}} + 1 \right) dx \right],
\end{aligned}$$

where the constant c depends only on n, m, γ_1, γ_2 and ε. Now choosing as usual $\varepsilon = \frac{\nu}{2cL}$ (note that we can also assume that $\varepsilon < 1 - 1/\gamma_2$ by choosing c large enough) we can absorb the first integral on the right hand side. Dividing the resulting inequality by $\nu/2$ leads us to

$$\int_{Q_{R/2}} |D^m u|^{p(x)}\, dx \le c \int_{Q_R} |F(x)|^{p(x)}\, dx + c \sum_{k=1}^{m} \int_{Q_R} \left(\frac{|D^{m-k}(u-P)|^{p_2}}{R^{kp_2}} + 1 \right) dx, \tag{8.12}$$

where c depends only on n, m, γ_1, γ_2 and L/ν. Taking into account the properties (8.9) of the polynomial P, using Poincaré's inequality and taking the mean values on both sides, we arrive at:

$$\fint_{Q_{R/2}} |D^m u|^{p(x)}\, dx \le c \fint_{Q_R} |F(x)|^{p(x)}\, dx + c \fint_{Q_R} \left(\left| \frac{D^{m-1}(u-P)}{R} \right|^{p_2} + 1 \right) dx.$$

Taking into account the definition of P, i.e. (8.9), we can apply Sobolev-Poincaré's inequality on the right hand side with exponents $p \equiv p_2$ and $p_* \equiv \frac{np_2}{n+p_2}$ to obtain

$$\fint_{Q_R} \left| \frac{D^{m-1}(u-P)}{R} \right|^{p_2} dx \le c_{SP} \left(\fint_{Q_R} |D^m u|^{\frac{np_2}{n+p_2}}\, dx \right)^{\frac{n+p_2}{n}}.$$

Note that the constants in Poincaré's and Sobolev-Poincaré's inequalities can be replaced by constants that only depend on γ_2 instead of p_2, thus $c_{SP} \equiv c(n, N, \gamma_2)$. Hölder's inequality, applied with the exponents $p \equiv \frac{p_1 s(n+p_2)}{p_2(n+1)} > 1$ provides (note $\tilde{s}^2 = \frac{n+1}{n}$)

$$\fint_{Q_R} |D^m u|^{\frac{np_2}{n+p_2}}\, dx \le \left(\fint_{Q_R} |D^m u|^{p_1/\tilde{s}}\, dx \right)^{\frac{p_2(n+1)}{p_1 \tilde{s}(n+p_2)}}.$$

Therefore we obtain

$$\fint_{Q_{R/2}} |D^m u|^{p(x)}\, dx \le c \fint_{Q_R} |F(x)|^{p(x)}\, dx + c \left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}}\, dx + 1 \right)^{\frac{p_2 \tilde{s}}{p_1}}.$$

Noting that $\frac{p_2\tilde{s}}{p_1} = \tilde{s} + \tilde{s}\left(\frac{p_2}{p_1} - 1\right)$ and $\frac{p_2}{p_1} - 1 \leq \frac{\omega(2nR)}{p_1}$ we get

$$\left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}} + 1\, dx\right)^{\frac{p_2\tilde{s}}{p_1}} \leq c\left[1 + \left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}}\, dx\right)^{\frac{\tilde{s}\omega(2nR)}{p_1}} \left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}}\, dx\right)^{\tilde{s}}\right],$$

where $c \equiv c(n, \gamma_1, \gamma_2)$. With $\frac{\tilde{s}}{p_1} \leq 2$ and $R \leq 1$ we obtain for the first term on the right hand side

$$\left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}}\, dx\right)^{\frac{\tilde{s}\omega(2nR)}{p_1}} \leq R^{-2n\omega(2nR)} \left(\int_{Q_R} |D^m u|^{p(x)}\, dx + |Q_R|\right)^{\frac{\tilde{s}\omega(2nR)}{p_1}}.$$

Noting that by the localization properties (8.1) we see that $R^{-2n\omega(2nR)} \leq c(n, \tilde{L})$ and taking into account the definition of K_0 we conclude the following reverse Hölder inequality (note also that $K_0 > 1$ and $p_1 \geq \gamma_1$):

$$\fint_{Q_{R/2}} |D^m u|^{p(x)}\, dx \leq c K_0^{\frac{2\omega(8nR_0)}{\gamma_1}} \left(\fint_{Q_R} |D^m u|^{p(x)/\tilde{s}}\, dx\right)^{\tilde{s}} + c\fint_{Q_R} \left(|F(x)|^{p(x)} + 1\right) dx, \tag{8.13}$$

where $K_0 > 1$ is from (8.2) and $c \equiv c\,(n, N, m, M, \gamma_1, \gamma_2, L/\nu)$. This inequality holds for any cube $Q_R \subseteq Q_{4R_0}$ and the appearing constants do not depend on the choice of the particular cube Q_R. Gehring's Lemma in the version which is written in [2] with $f \equiv |D^m u|^{p(x)/\tilde{s}}$ and $\varphi \equiv (|F|^{p(x)} + 1)^{1/\tilde{s}}$ under consideration of the restriction on σ finally provides the assertion. □

8.3. **Choice of constants and radii (II).** First we observe that, since $K_0 \geq 1$ (see the definition of K_0 in (8.2)), we have for any $K \geq K_0$:

$$\sigma_0 \geq \min\{1, q-1, c_g\}\, K^{-\frac{2q\omega(8nR_0)}{\gamma_1}}, \tag{8.14}$$

where σ_0 is the constant from Lemma 8.1. We set

$$K_M := \int_\Omega \left(|D^m u|^{p(x)} + |F|^{p(x)q} + 2\right) dx + 1, \tag{8.15}$$

and

$$\sigma_m := \min\left\{\frac{c_g}{K_M^{\frac{2q(\gamma_2-\gamma_1)}{\gamma_1}}}, \frac{q-1}{2}, 1\right\} > 0, \qquad \sigma_M := c_g + q. \tag{8.16}$$

Therefore $K_M \geq K_0$. Furthermore for any $1 \leq K \leq K_M$ we have

$$\sigma_m \leq \sigma_0 \leq \sigma_M. \tag{8.17}$$

We now choose the higher integrability exponent σ in Lemma 8.1 such that

$$\sigma := \tilde{\sigma}\sigma_0 \quad \text{with} \quad 0 < \tilde{\sigma} < \min\{\gamma_1 - 1, 1/2\}. \tag{8.18}$$

Then by (8.14) we have for any $\beta \in \left[\frac{\gamma_2}{\gamma_2-1}, \frac{\gamma_1}{\gamma_1-1}\right]$ and $K \geq K_0$:

$$\sigma^{-\beta} \leq c\tilde{\sigma}^{-\beta} K^{\beta\frac{2q\omega(8nR_0)}{\gamma_1}} \leq c\,(n, N, m, \gamma_1, \gamma_2, L/\nu, q)\, \tilde{\sigma}^{-\beta} K^{\frac{2q\omega(8nR_0)}{\gamma_1-1}}. \tag{8.19}$$

By the choice of σ in (8.18) and the structure of the constant σ_0 in Lemma 8.1 we have that

$$\sigma < \frac{q-1}{2}. \tag{8.20}$$

Now we impose for a fixed choice of $\tilde{\sigma}$ a further restriction on the size of R_0 by claiming

$$\max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \leq \frac{\tilde{\sigma}\sigma_m}{4}. \tag{8.21}$$

Therefore R_0 depends on $n, N, m, \gamma_1, \gamma_2, L/\nu, q, \||D^m u(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}, \||F(\cdot)|^{p(\cdot)}\|_{L^1(\Omega)}$ and $\tilde{\sigma}$. (8.21) immediately implies

$$\omega(8nR_0) \le \max\left\{2q\omega(8nR_0), \frac{2q\omega(8nR_0)}{\gamma_1 - 1}\right\} \le \frac{\tilde{\sigma}\sigma_m}{4} \le \frac{\tilde{\sigma}\sigma_0}{4} = \frac{\sigma}{4}. \tag{8.22}$$

8.4. **Calderón-Zygmund type estimate.** The key to the proof of Theorem 2.2 is the following lemma, which is an application of Lemma 5.1 to special sets X and Y.

Lemma 8.2. *Let $u \in W^{m,p(\cdot)}(\Omega;\mathbb{R}^N)$ be a weak solution of system (1.1) under the structure conditions (2.1), (2.2), (2.3) and (1.2). Furthermore let $\lambda \ge 1$, $0 < \tilde{\sigma} < 1$ as in (8.18) and $B_M > 1$. Then there exists a constant $A \equiv A(n, N, m, \gamma_1, \gamma_2, L/\nu, \tilde{L}) \ge 2$, independent of $\lambda, \tilde{\sigma}, u, A, F, B_M$ and a radius*

$$R_1 \equiv R_1(n, N, m, \gamma_1, \gamma_2, L/\nu, \tilde{L}, q, \tilde{\sigma}, B_M),$$

such that the following holds: If $R_0 \le R_1$ is so small that (8.1) and (8.21) hold, if K_0, σ_0 are the constants from (8.2), resp. Lemma 8.1, if $\sigma := \tilde{\sigma}\sigma_0$ as in (8.18), *if*

$$K := \int_{Q_{4R_0}} |D^m u|^{p(x)} + |F|^{p(x)(1+\sigma)}\, dx + 1, \tag{8.23}$$

if K_M and σ_M are the constants from (8.15), resp. (8.16), then for any $1 < B < B_M$ and $\vartheta \ge \left(B^{\frac{n(1+\delta)}{n-2}}\right)^{-1}$ there exists $\tilde{\varepsilon} > 0$, independent of λ, such that the following implication holds:

If on some $Q \in \mathcal{D}(Q_{R_0})$ we have

$$\left| Q \cap \left\{ x \in Q_{R_0} : M^*_{Q_{4R_0}}\left(|D^m u(\cdot)|^{p(\cdot)}\right)(x) > ABK^\sigma\lambda, \right.\right. \\ \left.\left. M^*_{1+\sigma, Q_{4R_0}}\left(|F(\cdot)|^{p(\cdot)} + 1\right)(x) < \tilde{\varepsilon}\lambda \right\}\right| > \vartheta |Q|, \tag{8.24}$$

then for the predecessor $\tilde{Q}$ of Q there holds

$$\tilde{Q} \subseteq \left\{ x \in Q_{R_0} : M^*_{Q_{4R_0}}\left(|D^m u(\cdot)|^{p(\cdot)}\right)(x) > \lambda \right\}. \tag{8.25}$$

■

Proof. We will prove the statement by contradiction. The constants $A, \tilde{\varepsilon}$ as well as the radius R_1 will be chosen at the end of the proof. Let us assume that (8.24) holds, but (8.25) is false. Then there exists a point $x_0 \in \tilde{Q}$, such that

$$M^*_{Q_{4R_0}}\left(|D^m u(\cdot)|^{p(\cdot)}\right)(x_0) \le \lambda,$$

i.e. we have

$$\fint_C |D^m u(x)|^{p(x)}\, dx \le \lambda, \tag{8.26}$$

for all cubes $C \subseteq Q_{4R_0}$ with $x_0 \in C$. We define $S := 2\tilde{Q}$. Since the cube $\tilde{Q}$ is obtained from the cube Q_{R_0} by at least one dyadic subdivision, we have $\tilde{Q} \subseteq Q_{R_0}$ and therefore $S \subseteq Q_{2R_0}$. Therefore by the smallness condition (8.1) imposed on the radius R_0 there holds

$$s := \operatorname{diam}(2S) \le 8nR_0, \quad \text{and therefore } \ \omega(s) \le \sigma/4. \tag{8.27}$$

In particular, since by $x_0 \in 2S$ the cube $2S \subseteq Q_{4R_0}$ is an admissible cube in the maximal function $M^*_{Q_{4R_0}}$, by (8.26) there holds

$$\fint_{2S} |D^m u(x)|^{p(x)}\, dx \le \lambda. \tag{8.28}$$

Additionally (8.24) implies

$$\left|\left\{ x \in Q : M^*_{1+\sigma, Q_{4R_0}}\left(|F(\cdot)|^{p(\cdot)} + 1\right)(x) < \tilde{\varepsilon}\lambda \right\}\right| > 0, \tag{8.29}$$

so that there exists at least one point $x \in Q$, in which the maximal function $M^*_{1+\sigma,Q_{4R_0}}$ of $|F(\cdot)|^{p(\cdot)}+1$ is small. Since $Q \subset 2S \subset Q_{4R_0}$, this implies

$$\left(\fint_S \left(|F|^{p(x)}+1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} < \tilde{\varepsilon}\lambda, \quad \left(\fint_{2S} \left(|F|^{p(x)}+1\right)^{1+\sigma} dx\right)^{\frac{1}{1+\sigma}} < \tilde{\varepsilon}\lambda. \tag{8.30}$$

We now use the localization argument from before in order to estimate $p(x)$ in a point x by constant exponents p_1, p_2. For this purpose we let

$$p_1 := p(x_m) = \min_{\overline{2S}} p(x), \quad p_2 := p(x_M) = \max_{\overline{2S}} p(x), \quad x_M, x_m \in \overline{2S}. \tag{8.31}$$

Obviously the exponents p_1 and p_2 depend on the local situation, especially on the cube $Q \in \mathcal{D}(Q_R)$. Thus in the following estimates it will be necessary to take care of the dependencies of the occurring constants on p_1 and p_2, eventually replacing them by constants which only depend on the global bounds γ_1 and γ_2 for p. We first use the fact that $2S \subseteq Q_{4R_0}$, obtaining by the restriction (8.21) on the radius R_0 for any $x \in \overline{2S}$

$$\begin{aligned} p_2 &= (p_2-p_1)+p_1 \le \omega\left(|x_M-x_m|\right)+p_1 \le \omega(s)+p_1 \le p_1\left(1+\omega(s)\right) \\ &\le p(x)\left(1+\omega(s)\right) \le p(x)\left(1+\omega(s)+\sigma/4\right) \le p(x)\left(1+\sigma\right). \end{aligned} \tag{8.32}$$

By (8.18) we have $\tilde{\sigma} < \gamma_1 - 1$. Recalling that $\sigma = \tilde{\sigma}\sigma_0$ and $\sigma_0 < 1$ (see Lemma 8.1) and $\gamma_1 \le p_1$ we have $\sigma \le p_1 - 1$, which implies

$$\begin{aligned} p_2\left(1+\sigma/4\right) &\le \left(p_1+\omega(s)\right)\left(1+\sigma/4\right) = p_1 + p_1\sigma/4 + \omega(s)\left(1+\sigma/4\right) \\ &\le p_1\left(1+\sigma/4+\omega(s)\right) \le p(x)\left(1+\sigma/4+\omega(s)\right) \le p(x)\left(1+\sigma\right). \end{aligned} \tag{8.33}$$

8.4.1. *Higher integrability.* First we note that the higher integrability from Lemma 8.1 together with (8.30) leads to an upper bound for the integral $\fint_S |D^m u|^{p_2}\, dx$. Since to prove this, we can follow line by line the estimates in [3, (59), p 132], we do not rewrite the estimates here. We obtain

$$u \in W^{m,p_2}(S), \tag{8.34}$$

and

$$\fint_S |D^m u|^{p_2}\ dx \le cK^{\sigma/4}\lambda, \qquad \int_S |D^m u|^{p_2}\ dx \le cK^{1+\sigma/4}, \tag{8.35}$$

where the constants depend only on $n, N, m, \gamma_1, \gamma_2, M$ and L/ν.

8.4.2. *The frozen system.* We consider the Dirichlet problem

$$\begin{cases} \displaystyle\int_S \left\langle A\left(x_M, D^m w\right), D^m \varphi\right\rangle\ dx = 0 & \text{for all } \ \varphi \in W_0^{m,p_2}(S;\mathbb{R}^N) \\ w \equiv u & \text{on } \partial S \end{cases}. \tag{8.36}$$

Since the vector field A is frozen in the point x_M, where the exponent $p(x_M) = p_2$ is constant, the structure conditions for the original vector field $A(x,z)$ (see (2.1) and the remark after (1.2)) lead to the following structure conditions for the frozen vector field $A(x_M, z)$:

$$\nu\left(\mu^2+|z|^2\right)^{(p_2-2)/2}|\lambda|^2 \le \langle D_z A(x_M,z)\lambda,\lambda\rangle \le L\left(\mu^2+|z|^2\right)^{(p_2-2)/2}|\lambda|^2, \tag{8.37}$$

$$|A\left(x_M,z\right)| \le L\left(\mu^2+|z|^2\right)^{(p_2-1)/2}, \tag{8.38}$$

for all $x \in S, z \in \mathbb{R}^{\mathcal{N}}$, where $\mu \in [0.1]$. It is easy to see that one can write these conditions also in the form

$$\nu\left(\mu^2+|z_1|^2+|z_2|^2\right)^{(p_2-2)/2}|z_2-z_1|^2 \le \left\langle A\left(x_M,z_2\right)-A\left(x_M,z_1\right), z_2-z_1\right\rangle, \tag{8.39}$$

$$\tfrac{\nu}{2}|z|^{p_2} \le \left\langle A\left(x_M,z\right), z\right\rangle + \nu\left((L/\nu)^{p_2}+1\right), \tag{8.40}$$

for all $z, z_1, z_2 \in \mathbb{R}^{\mathcal{N}}$.

The Dirichlet problem (8.36) admits a unique solution $w \in u + W_0^{m,p_2}(S;\mathbb{R}^N)$.

Since the vector field $A(x_M, z)$ fulfills the hypothesis of Lemma 3.1 (with p, Q_R replaced by p_2, S), we can apply the lemma in combination with Lemma 4.1 to conclude that

- in the case $p_2 \geq 2$ we have the estimate

$$\int_{\frac{3}{4}S} |DV_\mu(D^m w)|^{2(1+\tilde{\delta})}\, dx \leq \left(\frac{c}{R^2} \int_S |H_\mu(D^m w)|^2\, dx \right)^{1+\tilde{\delta}}, \tag{8.41}$$

  for any $\mu \in [0,1]$,
- in the case $1 < p_2 < 2$, for $\mu \in (0,1]$ there holds

$$\int_{\frac{3}{4}S} |DH_\mu(D^m w)|^{2(1+\tilde{\delta})}\, dx \leq \left(\frac{c}{R^2} \int_S |H_\mu(D^m w)|^2\, dx \right)^{1+\tilde{\delta}}, \tag{8.42}$$

  and for $\mu = 0$ we have

$$\int_{\frac{3}{4}S} |DV_0(D^m w)|^{2(1+\tilde{\delta})}\, dx \leq \left(\frac{c}{R^2} \int_S |H_0(D^m w)|^2\, dx \right)^{1+\tilde{\delta}}. \tag{8.43}$$

with $\tilde{\delta} \equiv \tilde{\delta}(n, m, N, \gamma_1, \gamma_2, L, \nu) > 0$. Here we used the abbreviations

$$V_\mu(D^m w) := \left(\mu^2 + |D^m w|^2\right)^{\frac{p_2-2}{4}} D^m w\,, \qquad H_\mu(D^m w) := \left(\mu^2 + |D^m w|^2\right)^{p_2/4}. \tag{8.44}$$

Note that the function $V_\mu : \mathbb{R}^k \to \mathbb{R}^k$ is quite common in the recent papers about regularity for systems and functionals. We will use here the following algebraic property of V_μ: For any $\mu \in [0,1], z, \eta \in \mathbb{R}^k$ there exists a constant $c \equiv c(n, \gamma_1, \gamma_2)$ such that

$$c^{-1}|z-\eta|(\mu^2 + |z|^2 + |\eta|^2)^{\frac{p_2-2}{4}} \leq |V_\mu(z) - V_\mu(\eta)| \leq c|z-\eta|(\mu^2 + |z|^2 + |\eta|^2)^{\frac{p_2-2}{4}}. \tag{8.45}$$

The proof of this property can be found for instance in [1, 17].

8.4.3. *Energy estimate.* Now we want to show the following energy estimate:

$$\fint_S |D^m w|^{p_2}\, dx \leq c\,(\gamma_2, L/\nu) \fint_S (|D^m u|^{p_2} + 1)\, dx. \tag{8.46}$$

To prove (8.46) we test (8.36) with $\varphi = u - w$. φ is an admissible test function, since $u, w \in W^{m,p_2}$ and $w - u \in W_0^{m,p_2}\left(S;\mathbb{R}^N\right)$. Using (8.40) we obtain

$$\begin{aligned} \nu/2 \int_S |D^m w|^{p_2}\, dx &\leq \int_S \langle A(x_M, D^m w), D^m w\rangle\, dx + \nu\left((L/\nu)^{p_2} + 1\right) \\ &= \int_S \langle A(x_M, D^m w), D^m u\rangle\, dx + \nu\left((L/\nu)^{p_2} + 1\right) \\ &\leq \int_S |A(x_M, D^m w)||D^m u|\, dx + \nu\left((L/\nu)^{p_2} + 1\right). \end{aligned}$$

The growth assumption (8.38) gives by Young's inequality

$$\begin{aligned} \int_S |A(x_M, D^m w)||D^m u|\, dx &\leq L \int_S \left(1 + |D^m w|^2\right)^{\frac{p_2-1}{2}} |D^m u|\, dx \\ &\leq \varepsilon 2^{p_2/2-1} \int_S |D^m w|^{p_2}\, dx + \varepsilon 2^{p_2/2-1} + \varepsilon^{1-p_2} L^{p_2} \int_S |D^m u|^{p_2}\, dx. \end{aligned}$$

Combining these estimates, the asserted estimate follows by choosing $\varepsilon = 2^{1-p_2/2}\nu/4$.

8.4.4. *Comparison estimate.* The next step is to establish a comparison estimate between $D^m u$ and $D^m w$. This turns out to be quite complicated, involving all the results from before, for example the energy estimate, higher integrability, the structure conditions of the frozen system, the localization and very fine estimates on the $L\log^\beta L$ scale. Nevertheless the argument and estimates are nearly the same as in the second order case and can be taken from [3, p 134ff]. Note that at this point the continuity assumption (2.2) comes into play. Although this condition differs from the condition in [3], the proof of the comparison estimate can be left unchanged. The reason for this is that we compare the original problem to a problem which is frozen in the point x_M where the growth exponent is the maximal exponent $p(x_M) = p_2$. Therefore in the application of (2.2) we pass over from the exponents $p(x)$ and $p(y)$ on the right hand side to this maximal value p_2. Since this is the only point where the continuity condition comes into play, there is no problem in replacing the continuity condition in [3] by our assumption (2.2).

Finally, one ends up with

$$
\begin{aligned}
(8.47)\qquad & \fint_S (\mu^2 + |D^m u|^2 + |D^m w|^2)^{\frac{p_2-2}{2}} |D^m u - D^m w|^2 \, dx \\
&\leq c_1 \omega(s) \log\left(\frac{1}{s}\right) K^\sigma \lambda + c_2 \omega(s) \tilde{\sigma}^{-1} K^\sigma \lambda + c_3 \tilde{\varepsilon}^{\frac{\gamma_2 - 1}{\gamma_2}} K^\sigma \lambda,
\end{aligned}
$$

with constants $c_1 \equiv c_1(n, N, m, \gamma_1, \gamma_2, L/\nu, M)$, $c_2 \equiv c_2(n, N, m, \gamma_1, \gamma_2, L/\nu, M, q)$ and $c_3 \equiv c_3(n, N, m, \gamma_1, \gamma_2, L/\nu)$.

8.4.5. *Estimate of the maximal function on level sets.* At this point of the proof we combine the a priori estimate for the solution of the frozen problem with the comparison estimate in order to estimate the super level sets of the maximal function of $|D^m u|^{p_2}$. We use Sobolev-Poincaré's inequality to translate the a priori higher differentiability of the solution of the frozen problem into higher integrability and therefore gain an exponent which we denote n_δ^* (see (8.50)). This exponent determines the decay of the super level sets of the maximal function on increasing levels (see (8.67)) and finally provides the desired higher integrability result.

We define the restricted maximal function to the cube $\frac{3}{2}\tilde{Q}$ by

$$M^{**} := M^*_{\frac{3}{2}\tilde{Q}},$$

whereas

$$M^* := M^*_{Q_{4R_0}}$$

denotes the maximal function on Q_{4R_0} (see the statement of Lemma 8.2).

For $1 < B < B_M$ we now estimate the measure of the set

$$\left\{ x \in Q : \; M^{**}(|D^m u|^{p_2})(x) > \tfrac{ABK^\sigma\lambda}{2}, \; M^*(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda \right\},$$

where A will be chosen later. First, by Lemma 7.1 we have:

$$
\begin{aligned}
|D^m u|^{p_2} &\leq (\mu^2 + |D^m u|^2)^{p_2/2} \\
(8.48)\qquad &\leq c_1 (\mu^2 + |D^m w|^2)^{p_2/2} + c_2 (\mu^2 + |D^m u|^2 + |D^m w|^2)^{\frac{p_2-2}{2}} |D^m u - D^m w|^2 \\
&= \tilde{c}_1 \mathcal{G}_1 + \tilde{c}_2 \mathcal{G}_2,
\end{aligned}
$$

where $\mathcal{G}_1 := (\mu^2 + |D^m w|^2)^{p_2/2}$, $\mathcal{G}_2 := (\mu^2 + |D^m u|^2 + |D^m w|^2)^{\frac{p_2-2}{2}} |D^m u - D^m w|^2$ and constants $\tilde{c}_1, \tilde{c}_2 \equiv \tilde{c}_1, \tilde{c}_2(n, \gamma_1, \gamma_2)$. Then there holds

$$
\begin{aligned}
&\left| \left\{ x \in Q : M^{**}(|D^m u|^{p_2})(x) > \tfrac{ABK^\sigma\lambda}{2}, \; M^*(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda \right\} \right| \\
&\quad \leq \left| \left\{ x \in Q : M^{**}(\mathcal{G}_1)(x) > \tfrac{ABK^\sigma\lambda}{2c_1}, \; M^*(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda \right\} \right| \\
&\qquad + \left| \left\{ x \in Q : M^{**}(\mathcal{G}_2)(x) > \tfrac{ABK^\sigma\lambda}{2c_2}, \; M^*(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}\lambda \right\} \right|
\end{aligned}
$$

$$=: I_1 + I_2.$$

Estimate for I_2: Using property (M1) for the maximal function , the inclusion $\frac{3}{2}\tilde{Q} = \frac{3}{4}S \subset S$ and the comparison estimate (8.47) we obtain

$$\begin{aligned} I_2 &\le \frac{c(n)c_2}{ABK^\sigma\lambda}\int_S (\mu^2 + |D^m u|^2 + |D^m w|^2)^{\frac{p_2-2}{2}} |D^m u - D^m w|^2\,dx \\ &\le \frac{c}{ABK^\sigma\lambda}\left(c_1\omega(s)\log\left(\frac{1}{s}\right)K^\sigma\lambda + c_2\omega(s)\tilde{\sigma}^{-1}K^\sigma\lambda + c_3\tilde{\varepsilon}^{\frac{\gamma_2-1}{\gamma_2}}K^\sigma\lambda\right)|S| \\ &= \frac{\hat{c}_1}{AB}\omega(s)\log\left(\frac{1}{s}\right)|S| + \frac{\hat{c}_2}{AB}\omega(s)\tilde{\sigma}^{-1}|S| + \frac{\hat{c}_3}{AB}\tilde{\varepsilon}^{\frac{\gamma_2-1}{\gamma_2}}|S|, \end{aligned}$$

with constants $\hat{c}_1 \equiv \hat{c}_1(n, N, m, \gamma_1, \gamma_2, L/\nu, M), , \hat{c}_2 \equiv \hat{c}_2(n, N, m, \gamma_1, \gamma_2, L/\nu, M, q)$ and $\hat{c}_3 \equiv \hat{c}_3(n, N, m, \gamma_1, \gamma_2, L/\nu)$.

Estimate for I_1: Since $\frac{n\chi}{n-\chi}$ is increasing in χ, we can find $\delta \equiv \delta(n, m, \gamma_1, \gamma_2, L/\nu) > 0$, such that

$$\frac{n}{n-2}(1+\delta) = \frac{(1+\tilde{\delta})n}{n-2(1+\tilde{\delta})}. \tag{8.49}$$

We set

$$r := \frac{n}{n-2}(1+\delta)p_2 = n_\delta^* p_2, \quad \text{with } n_\delta^* \equiv \frac{n}{n-2}(1+\delta), \tag{8.50}$$

and distinguish the cases $1 < p_2 < 2$ and $p_2 \ge 2$.

The case $p_2 \ge 2$: For $\eta \in \mathbb{R}^{\mathcal{N}}$ we estimate with (8.45) and Lemma 7.1 as follows (note that this estimate holds for any $p_2 > 1$)

$$\begin{aligned} \left(\mu^2 + |D^m w|^2\right)^{p_2/2} &\le c\left(\mu^2 + |\eta|^2\right)^{p_2/2} + c\left(\mu^2 + |D^m w|^2 + |\eta|^2\right)^{\frac{p_2-2}{2}}|D^m w - \eta|^2 \\ &\le c\left|V_\mu(D^m w) - V_\mu(\eta)\right|^2 + c\left(\mu^2 + |\eta|^2\right)^{p_2/2}, \end{aligned} \tag{8.51}$$

with the definition of $V_\mu(D^m w)$ of (8.44). Therefore by Corollary 6.2 we infer that (note also (8.49))

$$\begin{aligned} I_1 &\le \frac{c(n, r, p_2)c_1^{n_\delta^*}}{(ABK^\sigma\lambda)^{n_\delta^*}}|S| \fint_{\frac{3}{4}S}(\mu^2 + |D^m w|^2)^{\frac{p_2}{2}\frac{r}{p_2}}\,dx \\ &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\fint_{\frac{3}{4}S}|V_\mu(D^m w) - V_\mu(\eta)|^{\frac{2n(1+\tilde{\delta})}{n-2(1+\tilde{\delta})}}\,dx + \frac{c|S|(\mu^2+|\eta|^2)^{\frac{p_2 n(1+\delta)}{2(n-2)}}}{(ABK^\sigma\lambda)^{n_\delta^*}} \\ &=: I_{1,1}^{(1)} + I_{1,2}^{(1)}, \end{aligned}$$

with the obvious labelling of $I_{1,1}^{(1)}$ and $I_{1,2}^{(1)}$. We now choose η such that

$$V_\mu(\eta) = \fint_{\frac{3}{4}S} V_\mu(D^m w)\,dx. \tag{8.52}$$

Estimate for $I_{1,2}^{(1)}$: In the case $\mu = 0$ we have the identity

$$\left|(V_0(D^m w))_{\frac{3}{4}S}\right| = |V_0(\eta)| = |\eta|^{\frac{p_2-2}{2}}|\eta| = |\eta|^{p_2/2},$$

and therefore together with the energy estimate (8.46) and higher integrability (note that $\lambda \ge 1$, $K \ge 1$) a straight forward estimate shows

$$\begin{aligned} I_{1,2}^{(1)} &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(\fint_S |D^m w|^{p_2}\,dx\right)^{n_\delta^*} \\ &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(\fint_S (|D^m u|^{p_2} + 1)\,dx\right)^{n_\delta^*} \le \frac{c|S|}{(AB)^{n_\delta^*}}. \end{aligned}$$

In the case $\mu \in (0,1]$ we estimate (note that $p_2 \geq 2$):

$$|\eta|^{p_2} \leq |V_\mu(\eta)|^2 \leq \fint_{\frac{3}{4}S} |V_\mu(D^m w)|^2 \, dx \leq \fint_{\frac{3}{4}S} \left(\mu^2 + |D^m w|^2\right)^{p_2/2} \, dx,$$

and therefore with Hölder's inequality, the energy estimate (8.46) and higher integrability (note that $\mu \leq 1$) we easily deduce

$$\begin{aligned} I_{1,2}^{(1)} &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(1 + \fint_{\frac{3}{4}S} \left(\mu^2 + |D^m w|^2\right)^{p_2/2} \, dx\right)^{n_\delta^*} \\ &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(\fint_S (|D^m u|^{p_2} + 1) \, dx\right)^{n_\delta^*} \\ &\leq \frac{c|S|}{(AB)^{n_\delta^*}}. \end{aligned}$$

Estimate for $I_{1,1}^{(1)}$: By the choice of η we can apply Sobolev-Poincaré's inequality to obtain

$$\fint_{\frac{3}{4}S} |V_\mu(D^m w) - V_\mu(\eta)|^{\frac{2n(1+\tilde{\delta})}{n-2(1+\tilde{\delta})}} \, dx \leq c_{SP} \left(s^2 \fint_{\frac{3}{4}S} |DV_\mu(D^m w)|^{2(1+\tilde{\delta})} \, dx\right)^{n_\delta^*}.$$

The apriori estimate (8.41), taken together with the energy estimate (8.46) and again higher integrability now provides

$$I_{1,1}^{(1)} \leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(\fint_S |H_\mu(D^m w)|^2 \, dx\right)^{n_\delta^*} \leq \frac{c|S|}{(AB)^{n_\delta^*}}.$$

Taking all the estimates together we end up with

$$I_1 \leq \frac{c|S|}{(AB)^{n_\delta^*}},$$

where $c \equiv c(n, N, m, \gamma_1, \gamma_2, L/\nu, M)$.

The case $1 < p_2 < 2$: For $\mu \in (0,1]$ we first estimate by Corollary 6.2 (again note (8.49)):

$$\begin{aligned} I_1 &\leq \frac{c(n, r, p_2) c_1^{n_\delta^*}}{(ABK^\sigma \lambda)^{n_\delta^*}} |S| \fint_{\frac{3}{4}S} H_\mu(D^m w)^{\frac{2r}{p_2}} \, dx \\ &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \fint_{\frac{3}{4}S} \left|H_\mu(D^m w) - (H_\mu(D^m w))_{\frac{3}{4}S}\right|^{\frac{2n(1+\tilde{\delta})}{n-2(1+\tilde{\delta})}} \, dx + \frac{c|S| \left|(H_\mu(D^m w))_{\frac{3}{4}S}\right|^{n_\delta^*}}{(ABK^\sigma \lambda)^{n_\delta^*}} \\ &= I_{1,1}^{(2)} + I_{1,2}^{(2)}. \end{aligned}$$

with the definition for $H_\mu(D^m w)$ of (8.44) and the obvious labelling of $I_{1,1}^{(2)}$ and $I_{1,2}^{(2)}$.

Estimate for $I_{1,1}^{(2)}$: Applying Sobolev-Poincaré's inequality, the a priori estimate (8.42) and finally the energy estimate (8.46) and higher integrability, we obtain

$$\begin{aligned} I_{1,1}^{(2)} &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(s^2 \fint_{\frac{3}{4}S} |DH_\mu(D^m w)|^{2(1+\tilde{\delta})} \, dx\right)^{n_\delta^*} \\ &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(\fint_S |H_\mu(D^m w)|^2 \, dx\right)^{n_\delta^*} \\ &\leq \frac{c|S|}{(ABK^\sigma \lambda)^{n_\delta^*}} \left(\fint_S (|D^m u|^{p_2} + 1) \, dx\right)^{n_\delta^*} \leq \frac{c|S|}{(AB)^{n_\delta^*}}. \end{aligned}$$

Estimate for $I_{1,2}^{(2)}$: Here we use Hölder's inequality, the energy estimate (8.46) and higher integrability to conclude

$$\begin{aligned} I_{1,2}^{(2)} &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(\fint_S (|D^m w|^{p_2}+1)\,dx\right)^{n_\delta^*} \\ &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(\fint_S |D^m u|^{p_2}\,dx+1\right)^{n_\delta^*} \\ &\le \frac{c|S|}{(AB)^{n_\delta^*}}. \end{aligned}$$

In the case $\mu=0$ we proceed as follows, again using (8.51):

$$|D^m w|^{p_2} \le c\,|V_0(D^m w)-V_0(\eta)|^2 + c|\eta|^{p_2}.$$

Therefore we write by Corollary 6.2 and again noting (8.49):

$$\begin{aligned} I &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\fint_{\frac34 S}|D^m w|^{\frac{np_2(1+\delta)}{n-2}}\,dx \\ &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\fint_{\frac34 S}|V_0(D^m w)-V_0(\eta)|^{\frac{2n(1+\tilde\delta)}{n-2(1+\tilde\delta)}}\,dx + \frac{c|S||\eta|^{\frac{p_2 n(1+\delta)}{n-2}}}{(ABK^\sigma\lambda)^{n_\delta^*}} \\ &= I_{1,1}^{(3)} + I_{1,2}^{(3)}. \end{aligned}$$

We choose η such that

$$V_0(\eta) = \fint_{\frac34 S} V_0(D^m w)\,dx.$$

Estimate for $I_{1,1}^{(3)}$: Here we use Sobolev-Poincaré's inequality, the apriori estimate (8.43) and finally again the energy estimate (8.46) and higher integrability to conclude

$$\begin{aligned} I_{1,1}^{(3)} &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(s^2\fint_{\frac34 S}|DV_0(D^m w)|^{2(1+\tilde\delta)}\,dx\right)^{n_\delta^*} \\ &\le \frac{c|S|}{(ABK^\sigma\lambda)^{n_\delta^*}}\left(\fint_S |H_0(D^m w)|^2\,dx\right)^{n_\delta^*} \\ &\le \frac{c|S|}{(AB)^{n_\delta^*}}. \end{aligned}$$

Estimate for $I_{1,2}^{(3)}$: Since we have $|\eta|^{p_2/2} = |(V_0(D^m w))_{\frac34 S}|$, we can estimate $I_{1,2}^{(3)}$ in a completely analogous way as in the case $p_2\ge 2$ to obtain

$$I_{1,2}^{(3)} \le \frac{c|S|}{(AB)^{n_\delta^*}}.$$

Thus we have shown in any case (i.e. for any $p_2>1$ and for any $\mu\in[0,1]$):

$$I_1 \le \frac{\hat c_4|S|}{(AB)^{n_\delta^*}}, \tag{8.53}$$

with $\hat c_4 \equiv \hat c_4(n,N,m,\gamma_1,\gamma_2,L/\nu)$. Connecting this with the estimate valid for I_2 we finally arrive at (eventually enlarging the constants by a factor $c(n)$)

$$\begin{aligned} &|\{x\in Q : M^{**}(|D^m u|^{p_2})(x) > \tfrac{AB}{2}K^\sigma\lambda,\ M^*(|F(\cdot)|^{p(\cdot)}+1)(x)<\tilde\varepsilon\lambda\}| \\ &\quad\le \Big[\frac{\hat c_1}{AB}\omega(s)\log\left(\tfrac1s\right) + \frac{\hat c_2}{AB}\omega(s)\tilde\sigma^{-1} + \frac{\hat c_3}{AB}\tilde\varepsilon^{\frac{\gamma_2-1}{\gamma_2}} + \frac{\hat c_4}{(AB)^{n_\delta^*}}\Big]|Q|. \end{aligned}$$

Now we come to the rather involved choice of the parameters. First we determine $R_1 \equiv R_1(n, N, m, \gamma_1, \gamma_2, L/\nu, \omega(\cdot), \tilde{\sigma}, A, B_M)$ small enough to have

$$\frac{\hat{c}_1}{A}\omega(s)\log\left(\tfrac{1}{s}\right) \leq \frac{1}{8B_M^{n_\delta^*-1}} \qquad \text{and} \qquad \frac{\hat{c}_2}{A}\omega(s)\tilde{\sigma}^{-1} \leq \frac{1}{8B_M^{n_\delta^*-1}},$$

for all $s \leq 8nR_1$. Then if $R_0 \leq R_1$ satisfies (8.1) and (8.21), we have

$$R_0 \equiv R_0(n, N, m, \gamma_1, \gamma_2, \nu, L, \||D^m u(\cdot)|^{p(\cdot)}\|_{L^1}, \||F(\cdot)|^{p(\cdot)}\|_{L^{\frac{n}{n-\gamma_1}}}, \omega(\cdot), B_M).$$

Next we choose $\tilde{\varepsilon} \equiv \tilde{\varepsilon}(n, N, m, \gamma_1, \gamma_2, \nu, L, A, B) \in (0,1)$ such that

$$\frac{\hat{c}_3}{A}\tilde{\varepsilon}^{\frac{\gamma_2-1}{\gamma_2}} = \frac{1}{8B^{n_\delta^*-1}}. \tag{8.54}$$

Next we fix A by

$$A = \max\{(8\hat{c}_4)^{n_\delta^*}, 5^{n+1}\} \geq 2, \tag{8.55}$$

which yields

$$\frac{\hat{c}_4}{(AB)^{n_\delta^*}} \leq \frac{1}{8B^{n_\delta^*}}.$$

Noting that

$$\frac{1}{8B_M^{n_\delta^*-1}} \leq \frac{1}{8B^{n_\delta^*-1}},$$

for $1 < B \leq B_M$ we obtain for any $R \leq R_0$ that

$$|\{x \in Q : \ M^{**}(|D^m u|^{p_2})(x) > \tfrac{AB}{2}K^\sigma\lambda, \ M^*(|F(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\}| \leq \frac{|Q|}{2B^{n_\delta^*}}.$$

In particular for every ϑ satisfying $\vartheta \geq \frac{1}{B^{n_\delta^*}}$ there holds

$$|\{x \in Q : \ M^{**}(|D^m u|^{p_2})(x) > \tfrac{AB}{2}K^\sigma\lambda, \ M^*(|F(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\}| \leq \tfrac{\vartheta}{2}|Q|. \tag{8.56}$$

We next want to turn this estimate for the maximal function with the fixed exponent p_2 into an estimate for the maximal function of $|D^m u|^{p(\cdot)}$. Since $p_2 \geq p(x)$ for any $x \in 2\tilde{Q}$, we see that for any cube $Q \subset \frac{3}{2}\tilde{Q}$ we have

$$⨍_Q |D^m u|^{p(x)}\,dx \leq ⨍_Q |D^m u|^{p_2}\,dx + 1.$$

Hence, for $x \in Q$ there holds

$$M^{**}(|D^m u|^{p(\cdot)})(x) \leq M^{**}(|D^m u|^{p_2}+1)(x).$$

Since $\lambda, K, \frac{A}{2}, B \geq 1$, we have in particular that $\frac{AB}{2}K^\sigma\lambda \geq 1$ and therefore

$$M^{**}(|D^m u(\cdot)|^{p(\cdot)})(x) > ABK^\sigma\lambda$$

implies

$$M^{**}(|D^m u|^{p_2})(x) + \tfrac{AB}{2}K^\sigma\lambda \geq M^{**}(|D^m u|^{p_2}+1)(x) > ABK^\sigma\lambda = 2\cdot\tfrac{AB}{2}K^\sigma\lambda.$$

From (8.56) we therefore obtain

$$|\{x \in Q : M^{**}(|D^m u(\cdot)|^{p(\cdot)})(x) > ABK^\sigma\lambda, M^*(|F(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\}| \leq \tfrac{\vartheta}{2}|Q|. \tag{8.57}$$

The last step in the proof consists in converting (8.57) into an estimate for the restricted maximal function $M^* = M^*_{Q_{4R_0}}$. This can be achieved by looking carefully at the cubes involved in the proof. Let ℓ be the sidelength of the cube Q. For an arbitrary point $x \in Q$ both x itself and the point x_0 chosen in (8.26) are contained in the cube $\tilde{Q}$ which has sidelength 2ℓ.
Now if $C' \subseteq Q_{4R_0}$ is a cube, containing x and having side length ℓ' larger than $\ell/2$, there holds $C' \cap \tilde{Q} \neq \emptyset$. Thus there exists a cube $C'' \subseteq Q_{4R_0}$, containing C' and $\tilde{Q}$, and whose side length ℓ'' is bounded by

$$\ell'' \leq 2\ell + \ell' \leq 5\ell'.$$

Therefore, by (8.26) there holds

$$\fint_{C'} |D^m u|^{p(x)}\,dx \le \frac{1}{|C'|}\fint_{C''} |D^m u|^{p(x)}\,dx \le \frac{|C''|}{|C'|}\lambda \le 5^n\lambda,$$

while in the case $\ell' \le \frac{\ell}{2}$, we have $C' \subset \frac{3}{2}\tilde{Q}$ and

$$\fint_{C'} |D^m u|^{p(x)}\,dx \le M^{**}(|D^m u|^{p(\cdot)})(x).$$

This implies that

$$M^*(|D^m u|^{p(\cdot)})(x) \le \max\Big\{M^{**}(|D^m u|^{p(\cdot)})(x), 5^n\lambda\Big\} \qquad \text{for all } \ x \in Q.$$

From the choice of A, i.e. (8.55) we infer that $\frac{AB}{2}K^\sigma \ge \frac{5^{n+1}}{2} > 5^n$.

$$\Big\{x \in Q : M^*(|D^m u(\cdot)|^{p(\cdot)})(x) > ABK^\sigma\lambda\Big\} \subseteq \Big\{x \in Q : M^{**}(|D^m u(\cdot)|^{p(\cdot)})(x) > ABK^\sigma\lambda\Big\},$$

and therefore

$$|\{x \in Q : M^*(|D^m u(\cdot)|^{p(\cdot)})(x) > ABK^\sigma\lambda, M^*(|F(\cdot)|^{p(\cdot)}+1)(x) < \tilde{\varepsilon}\lambda\}| \le \tfrac{\vartheta}{2}|Q|.$$

This contradicts (8.24) and completes the proof of Lemma 8.2. □

8.5. **Proof of the main theorem.** We now apply Lemma 8.2 in order to obtain the result of the main theorem. Since the procedure follows the one of [3, pp 141-146], we only sketch the main steps here, nevertheless explicitely pointing out the special choice of the constants and parameters.

First we define

$$\mu_1(t) := \Big|\Big\{x \in Q_{R_0} : M^*(|D^m u(\cdot)|^{p(\cdot)})(x) > t\Big\}\Big|,$$

$$\mu_2(t) := \Big|\Big\{x \in Q_{R_0} : M^*_{1+\sigma}(|F(\cdot)|^{p(\cdot)}+1)(x) > t\Big\}\Big|,$$

with $M^* \equiv M^*_{Q_{4R_0}}$. For $q \in (1, \frac{n}{n-2})$ we set

$$B_M := (2(AK_M^{\sigma_M})^q)^{\frac{n-2}{n(1+\delta)-q(n-2)}} = B_M(n,q,\delta,K_M,\sigma_M). \tag{8.58}$$

By the restriction imposed on the range of q there holds $\frac{n-2}{n(1+\delta)-q(n-2)} > \frac{n-2}{n\delta+2} > 0$ and therefore, since $A \ge 2, K_M > 1, \sigma_M > 0$, we have $B_M > 1$. With this choice of B_M we set for $1 < K < K_M$ and $0 < \sigma < \sigma_M$:

$$B := (2(AK^\sigma)^q)^{\frac{n-2}{n(1+\delta)-q(n-2)}}. \tag{8.59}$$

Then

$$1 < B < B_M,$$

and moreover

$$B^{-n^*_\delta+q} = \frac{1}{2(AK^\sigma)^q}. \tag{8.60}$$

Now, we let

$$\lambda_0 := \frac{5^{n+2}c_W}{\vartheta}\fint_{Q_{4R_0}} |D^m u|^{p(x)}\,dx + 1, \tag{8.61}$$

where we have chosen

$$\vartheta := B^{-n^*_\delta}. \tag{8.62}$$

Here $c_W \equiv c_W(n)$ denotes the constant from (6.3). By (6.3) and the definition of λ_0 we obtain

$$\mu_1(\lambda_0) \le \frac{c_W}{\lambda_0}\fint_{Q_{4R_0}} |D^m u|^{p(x)}\,dx \cdot |Q_{4R_0}| \le \frac{4^n|Q_{R_0}|\vartheta}{5^{n+2}} \le \tfrac{\vartheta}{2}|Q_{R_0}|. \tag{8.63}$$

Let A be the constant from Lemma 8.2. Since $A, B, K \geq 1$ we have $ABK^\sigma \geq 1$ and therefore

$$\mu_1\left((ABK^\sigma)^h \lambda_0\right) \leq \tfrac{\vartheta}{2}|Q_{R_0}| \qquad \text{for} \quad h \in \mathbb{N} \cup \{0\}. \tag{8.64}$$

We now set

$$\tilde{A} := ABK^\sigma \geq 2. \tag{8.65}$$

It can easily be checked that with $h \in \mathbb{N} \cup \{0\}$ the assumptions of Lemma 8.2 are satisfied for the sets

$$X := \{x \in Q_{R_0} : M^*(|D^m u(\cdot)|^{p(\cdot)})(x) > \tilde{A}^{h+1}\lambda_0, M^*_{1+\sigma}(|F(\cdot)|^{p(\cdot)} + 1)(x) < \tilde{\varepsilon}(AB)^h \lambda_0\}$$

and

$$Y := \{x \in Q_{R_0} : \ M^*(|D^m u(\cdot)|^{p(\cdot)})(x) > \tilde{A}^h \lambda_0\},$$

which provides the estimate

$$\mu_1(\tilde{A}^{h+1}\lambda_0) \leq B^{-n^*_\delta}\mu_1(\tilde{A}^h\lambda_0) + \mu_2(\tilde{\varepsilon}\tilde{A}^h\lambda_0) \tag{8.66}$$

Iterating this inequality and exploiting the definition of A and the specific choice of B we obtain for arbitrary $J \in \mathbb{N}$

$$\mu_1(\tilde{A}^{h+1}\lambda_0) \leq \mu_1(\lambda_0) + \tilde{A}^q \sum_{i=0}^{J} \mu_2(\tilde{\varepsilon}\tilde{A}^i\lambda_0)\tilde{A}^i \sum_{k=0}^{J-i} \tilde{A}^{qk} B^{-n^*_\delta k}. \tag{8.67}$$

Using the choice of B and the definition of $\tilde{A}$ we infer that $\tilde{A}^{qk}B^{-n^*_\delta k} = ((AK^\sigma)^q B^{q-n^*_\delta})^k = 2^{-k}$. Hence, the last sum on the right hand side above can be uniformly estimated from above by

$$\sum_{k=0}^{J-i} \tilde{A}^{qk} B^{n^*_\delta k} \leq 2.$$

Since the estimate holds for any $J \in \mathbb{N}$, we obtain, passing to the limit $J \to \infty$:

$$\sum_{k=1}^{\infty} \tilde{A}^{qk}\mu_1(\tilde{A}^k\lambda_0) \leq \mu_1(\lambda_0) + 2\tilde{A}^q \sum_{k=0}^{\infty} \tilde{A}^{kq}\mu_2(\tilde{A}^k\lambda_0\tilde{\varepsilon}). \tag{8.68}$$

This estimate can be transformed into an estimate for the maximal function. Applying the elementary identity

$$\int_Q g^q \, dx = \int_0^\infty q\lambda^{q-1} \left|\{x \in Q : \ g(x) > \lambda\}\right| d\lambda,$$

which holds for $g \in L^q(Q), g \geq 0$, $q \geq 1$, to the maximal function of $|D^m u|^{p(\cdot)}$, decomposing the inteval $[0, \infty)$ into intervals $[0, \lambda_0]$ and $[\tilde{A}^n\lambda_0, \tilde{A}^{n+1}\lambda_0]$ and exploiting (8.68) in combination with the monotonicity of $\mu_1(t)$ and $\mu_2(t)$ and finally using the L^p estimate for the maximal function, we end up with the estimate

$$\int_{Q_{R_0}} |D^m u|^{p(x)q} \, dx \leq |Q_{R_0}|\lambda_0^q + 2(\tilde{A}\lambda_0)^q\mu_1(\lambda_0) + c(n)\frac{q^2}{q-1} \cdot \frac{\tilde{A}^{2q}}{\tilde{\varepsilon}^q} \int_{Q_{4R_0}} |F|^{p(x)q} + 1 \, dx. \tag{8.69}$$

By (8.63), by the choice of ϑ in (8.62) and (8.60) we obtain

$$2(\tilde{A}\lambda_0)^q\mu_1(\lambda_0) \leq (\tilde{A}\lambda_0)^q B^{-n^*_\delta}|Q_{R_0}| \overset{(8.60)}{=} (AK^\sigma)^q\lambda_0^q \frac{1}{2(AK^\sigma)^q}|Q_{R_0}| = \tfrac{1}{2}\lambda_0^q|Q_{R_0}|.$$

Furthermore, recalling the definition of λ_0 in (8.61) and the choice of ϑ, and taking into account (8.59), as well as the dependencies of the constant A (recall also the definition of n^*_δ in (8.50), we see that

$$\lambda_0 = c(n)B^{n^*_\delta} \fint_{Q_{4R_0}} \left(|D^m u|^{p(x)} + 1\right) dx = cK^{\frac{\sigma n(1+\delta)q}{n(1+\delta)-q(n-2)}} \fint_{Q_{4R_0}} \left(|D^m u|^{p(x)} + 1\right) dx,$$

with $c \equiv c(n, \gamma_1, \gamma_2, \nu, L, M)$. To treat the last term appearing on the right hand side of (8.69) we need to control $\tilde{A}^{2q}/\tilde{\varepsilon}^q$. Recalling the definitions of $\tilde{\varepsilon}, \tilde{A}$ and B from (8.54), (8.65) and (8.59) we find that

$$\frac{\tilde{A}^2}{\tilde{\varepsilon}} = (ABK^\sigma)^2 \left(\frac{A}{8B^{\frac{n(1+\delta)}{n-2}-1}} \cdot \frac{1}{\hat{c}_3} \right)^{-\frac{\gamma_1}{\gamma_1-1}} = cK^{\frac{2q\sigma(n-2)}{n(1+\delta)-q(n-2)} + \frac{2q\gamma_1\sigma}{(n(1+\delta)-q(n-2))(\gamma_1-1)} + 2\sigma},$$

where $c \equiv c(n, \gamma_1, \gamma_2, L/\nu, M, q)$. Thus passing to the averages leeds to

$$\begin{aligned}\left(\fint_{Q_{R_0}} |D^m u|^{p(x)q}\, dx \right)^{1/q} \quad &\leq \quad cK^{\frac{\sigma n(1+\delta)q}{n(1+\delta)-q(n-2)}} \fint_{Q_{4R_0}} |D^m u|^{p(x)} + 1\, dx \\ &\quad + cK^{\frac{2(n-2)q\sigma}{n(1+\delta)-q(n-2)} + \frac{2q\gamma_1\sigma}{(n(1+\delta)-q(n-2))(\gamma_1-1)} + 2\sigma} \left(\fint_{Q_{4R_0}} |F|^{p(x)q} + 1\, dx \right)^{1/q}.\end{aligned}$$

For given $\varepsilon > 0$, we now want to reach the following smallness conditions to be fulfilled:

$$\frac{\sigma n(1+\delta)q}{n(1+\delta)-q(n-2)} \leq \varepsilon \tag{8.70}$$

and

$$\frac{2(n-2)\sigma q}{n(1+\delta)-q(n-2)} \leq \frac{\varepsilon}{3}, \quad \frac{2\gamma_1\sigma q}{(n(1+\delta)-q(n-2))(\gamma_1-1)} \leq \frac{\varepsilon}{3}, \quad 2\sigma \leq \frac{\varepsilon}{3}. \tag{8.71}$$

These conditions hold for example, if

$$\sigma \leq \frac{\varepsilon}{9} \min\left\{ 1, \frac{(n(1+\delta)-q(n-2))(\gamma_1-1)}{q\gamma_1}, \frac{n(1+\delta)-q(n-2)}{n(1+\delta)q} \right\}. \tag{8.72}$$

Since we have (choosing δ small enough) $\frac{(n(1+\delta)-q(n-2))(\gamma_1-1)}{q\gamma_1} < \frac{n(1+\delta)-q(n-2)}{q} < \frac{n(1+\delta)}{q} - n + 2 < 3$, and $\frac{n(1+\delta)-q(n-2)}{nq} < \frac{2}{n} + \delta < 3$, (8.72) implies $\sigma < \frac{\varepsilon}{3}$. To reach this, we set

$$\bar{\sigma} := \frac{\varepsilon}{9\sigma_M} \min\left\{ 1, \frac{(n(1+\delta)-q(n-2))(\gamma_1-1)}{q\gamma_1}, \frac{n(1+\delta)-q(n-2)}{n(1+\delta)q} \right\}.$$

and finally

$$\tilde{\sigma} = \min\left\{ \bar{\sigma}, \gamma_1 - 1, \tfrac{1}{2} \right\}. \tag{8.73}$$

Thus $\tilde{\sigma} \equiv \tilde{\sigma}(n, q, \gamma_1, \varepsilon, c_g, \delta)$ is fixed. With this choice, there holds (note that $\sigma_0 < \sigma_M$ and the estimates above):

$$\sigma = \tilde{\sigma}\sigma_0 = \frac{\varepsilon\sigma_0}{9\sigma_M} \min\left\{ 1, \frac{(n(1+\delta)-q(n-2))(\gamma_1-1)}{q\gamma_1}, \frac{n(1+\delta)-q(n-2)}{n(1+\delta)q} \right\} < \frac{\varepsilon}{3} < \varepsilon.$$

Then $\sigma \leq \varepsilon$ and $\varepsilon < q-1$ implies (note also that $|Q_{4R_0}| \leq 1$ since $8nR_0 \leq 1$):

$$K = \int_{Q_{4R_0}} \left(|D^m u|^{p(x)} + |F|^{p(x)(1+\sigma)} \right) dx + 1 \leq \int_{Q_{4R_0}} \left(|D^m u|^{p(x)} + |F|^{p(x)(1+\varepsilon)} \right) dx + 2.$$

Remark on the dependencies of the constants: By the choice of $\tilde{\sigma} = \tilde{\sigma}(\varepsilon)$, also $R_1 \equiv R_1(\tilde{\sigma}) = R_1(\varepsilon)$ is fixed via Lemma 8.2, and then also $R_0 \equiv R_0(\varepsilon)$ via Lemma 8.2 and (8.21).

Therefore for any cube Q_R with $R \leq R_0$, $Q_{4R} \Subset \Omega$ there holds

$$\left(\fint_{Q_R} |D^m u|^{p(x)q}\, dx \right)^{1/q} \leq cK^\varepsilon \fint_{Q_{4R}} |D^m u|^{p(x)} + 1\, dx + cK^\varepsilon \left(\fint_{Q_{4R}} |F|^{p(x)q} + 1\, dx \right)^{1/q},$$

in which the constant depends on $n, N, m, \gamma_1, \gamma_2, L/\nu$ and q, and with

$$K = \int_{Q_{4R}} \left(|D^m u|^{p(x)} + |F|^{p(x)(1+\varepsilon)} \right) dx + 1.$$

Therefore the statement $|D^m u|^{p(\cdot)} \in L^q_{loc}(\Omega)$ follows by a covering argument. $\boxed{QED}$

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Jens Habermann, Institute for mathematics, Friedrich-Alexander University, Bismarckstr. 1 1/2, 91054 Erlangen, Germany;

E-mail address: habermann@mi.uni-erlangen.de